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Revealing hidden steering nonlocality in a quantum network

Kaushiki Mukherjee[1,a], Biswajit Paul[2,b], and Soma Mandal[1,c]

[1] Department of Mathematics, Government Girls' General Degree College, Ekbalpore, Kolkata 700023, India
[2] Department of Mathematics, Balagarh Bijoykrishna Mahavidyalaya, Balagarh, Hooghly 712501, India

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Abstract. By combining two objects with no quantum effect one can get an object with quantum effect. Such a phenomenon, often referred to as *activation*, has been analyzed for the notion of steering nonlocality. Activation of steering nonlocality is observed for different classes of mixed entangled states in linear network scenarios. Characterization of arbitrary two qubit states, in ambit of steering activation in network scenarios has been provided in this context. Using the notion of reduced steering, instances of steerability activation are also observed in nonlinear network. Present analysis involves three measurement settings scenario (for both trusted and untrusted parties) where steering nonlocality is distinguishable from Bell nonlocality.

1 Introduction

Quantum nonlocality is an inherent feature of quantum theory [1,2]. It forms the basis of various information theoretic tasks [3–10]. Presence of entanglement is a necessary condition for generation of nonlocal correlations, though it is not sufficient due to existence of local models of some mixed entangled states [11–13]. Such type of entangled states are often referred to as *local entangled states* [14]. Procedures involving exploitation of nonlocal correlations from local entangled states are often referred to as *activation scenarios.* [15]. Till date, such activation scenarios are classified into three categories: *activation via local filtering* [16–18], *activation by tensoring* [19–23] and *activation in quantum networks.*. Any possible combination of mechanisms involved in these three types is also considered as a valid activation procedure.

In activation by quantum network scenarios, nonlocality is activated by suitable arrangement of states (different or identical copies) in a quantum network [24–28]. Speaking of the role of quantum networks in activation, entanglement swapping networks have emerged as a useful tool for activating nonlocality of states in standard Bell scenario. In present discussion, utility of these networks will be explored in ambit of activating nonlocality beyond Bell scenario.

In an entanglement swapping network, entanglement is created between two distant parties sharing no direct common past [29–31]. Apart from its fundamental importance, it is applicable in various quantum applications. This procedure is also a specific example of quantum teleportation [32].

The key point of quantum nonlocality activation (Bell-CHSH sense) in entanglement swapping scenario is that starting from entangled states (shared between interacting parties) satisfying Bell-CHSH inequality, a Bell-nonlocal state is generated between non-interacting parties at the end of the protocol. In [24,27,28] swapping procedure has been framed as a novel example of nonlocality activation in quantum mechanics. Existing research works have exploited bipartite [24,27,28] and tripartite hidden nonlocality [33] in standard Bell scenario using swapping network. Present work will be exploring the utility(if any) of entanglement swapping network for activation of quantum steering nonlocality. Owing to involvement of sequential measurements in the network scenario, we will refer activation of steering nonlocality as *revealing hidden steering nonlocality* in spirit of Popescu [16].

Motivated by famous EPR argument [1] claiming incompleteness of quantum theory, Schrodinger first gave the concept of *steering* [34,35]. A complete mathematical formalism of such a manifestation of steering was provided in [36] where they characterized *steering correlations*. Several criteria have emerged for detecting steerability of correlations generated from a given quantum state [37–47]. The correlation-based criterion given in [39], often referred to as CJWR inequality, is used here for analyzing activation of steerability. Up to two measurement settings scenario, notions of Bell-CHSH nonlocality and any steering nonlocality are indistinguishable. So, here we consider CJWR inequality for three measurement settings. Violation of this symmetric inequality guarantees steerability of the bipartite

[a] e-mail: kaushiki.wbes@gmail.com (corresponding author)
[b] e-mail: biswajitpaul4@gmail.com
[c] e-mail: soma2778@gmail.com

correlations generated in the corresponding measurement scenario. Such form of steerability is often referred to as F_3 *steerability*. Using such a symmetric inequality as a detection criterion allows interchange of the roles of the trusted and the untrusted parties in the operational interpretation of steering.

Now consider a scenario involving two entangled states (ρ_{AB}, ρ_{BC},say) such that none of them violates CJWR inequality for three settings [39]. Let ρ_{AB} and ρ_{BC} be shared between three distant parties Alice, Bob and Charlie(say) where Alice and Charlie share no direct common past. Let ρ_{AB} be shared between Alice and Bob(say), whereas ρ_{BC} be shared between Bob and Charlie. Let classical communication be allowed between two parties sharing a state. Hence, Alice and Charlie do not interact. In such a scenario, when the parties perform local operations, will it be possible to generate a steerable state between the two non-interacting parties? Affirmative result is obtained when one considers an entanglement swapping network. To be precise, for some outputs of Bob, conditional state shared between the two non-interacting parties (Alice and Charlie) turns out to be F_3 steerable.

After observing hidden steerability for some families of two qubit states in a standard entanglement swapping network (Fig. 1), a characterization of arbitrary two qubits states is given in this context. As already mentioned before, CJWR inequality (for three settings) given in [39] is used as a detection criterion. Instance of genuine activation of steering is also observed in the sense that steerable state is obtained while using unsteerable states in the swapping protocol. Arbitrary two qubit states have also been characterized in perspective of genuine activation. At this junction it should be pointed out that the steerable conditional states resulting at the end of the protocol are Bell-local in corresponding measurement scenario [48].

Exploring hidden steerability in three party entanglement swapping scheme, number of parties is then increased. Results of activation are observed in a star network configuration of entanglement swapping involving nonlinear arrangement of four parties under some suitable measurement contexts.

Rest of our work is organized as follows. In Sect. 2, we provide the motivation underlying present discussion. In Sect. 3, we provide with some mathematical preliminaries. Activation of steerability in three party network scenario is analyzed in Sect. 4. In next section, revelation of hidden steerability is then discussed when number of parties is increased in a nonlinear fashion (in Sect. 5). Phenomenon of genuine activation of steering nonlocality is discussed in Sect. 6 followed by concluding remarks in Sect. 7.

2 Motivation

Steerable correlations are used in various quantum information processing tasks such as cryptography [49–54], randomness certification [55–59], channel discrimination [60,61] and many others. So any steerable quantum state is considered a useful resource. Though pure entangled states are best candidate in this context, but these are hardly available. Consequently, mixed entangled states are used in practical situations all of which are not steerable. From practical perspectives, exploiting steerability from unsteerable entangled states thus warrants attention. In this context revelation of hidden steerability from unsteerable quantum states basically motivates present discussion. Choosing network scenario based on entanglement swapping for the activation purpose is further motivated by the fact that steerable correlations can be generated between two non-interacting parties once the states involved are subjected to suitable LOCC [62]. Such nonclassical correlations in turn may be used as a resource in network-based quantum information and communication protocols [63–65].

3 Preliminaries

3.1 Bloch vector representation

Let ϱ denotes a two qubit state shared between two parties.

$$\varrho = \frac{1}{4}(\mathbb{I}_{2\times 2} + \vec{\mathfrak{u}}.\vec{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \vec{\mathfrak{v}}.\vec{\sigma} + \sum_{j_1,j_2=1}^{3} w_{j_1 j_2} \sigma_{j_1} \otimes \sigma_{j_2}), \tag{1}$$

with $\vec{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$, σ_{j_k} denoting Pauli operators along three mutually perpendicular directions($j_k = 1, 2, 3$). $\vec{\mathfrak{u}} = (l_1, l_2, l_3)$ and $\vec{\mathfrak{v}} = (r_1, r_2, r_3)$ stand for the local Bloch vectors($\vec{\mathfrak{u}}, \vec{\mathfrak{v}} \in \mathbb{R}^3$) of party $\mathcal{A}$ and $\mathcal{B}$, respectively, with $|\vec{\mathfrak{u}}|, |\vec{\mathfrak{v}}| \leq 1$ and $(w_{i,j})_{3\times 3}$ denotes the correlation tensor $\mathcal{W}$(a real matrix). The components $w_{j_1 j_2}$ are given by $w_{j_1 j_2} = \mathrm{Tr}[\rho\, \sigma_{j_1} \otimes \sigma_{j_2}]$.

On applying suitable local unitary operations, the correlation tensor becomes diagonalized:

$$\varrho^{'} = \frac{1}{4}(\mathbb{I}_{2\times 2} + \vec{\mathfrak{a}}.\vec{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \vec{\mathfrak{b}}.\vec{\sigma} + \sum_{j=1}^{3} \mathfrak{t}_{jj} \sigma_j \otimes \sigma_j), \tag{2}$$

Here, the correlation tensor is $T = \mathrm{diag}(t_{11}, t_{22}, t_{33})$. Under local unitary operations entanglement content of a quantum state remains invariant. Hence, steerability of ϱ and $\varrho^{'}$ remain the same.

3.2 Steering inequality

A linear steering inequality was derived in [39]. Under the assumption that both the parties sharing a bipartite state(ρ_{AB}) perform n dichotomic quantum measurements (on their respective particles), Cavalcanti, Jones, Wiseman, and Reid(CJWR) formulated a series of correlators-based inequalities [39] for checking steerability of ρ_{AB}:

$$\mathcal{F}_n(\rho_{AB}, \nu) = \frac{1}{\sqrt{n}} |\sum_{l=1}^{n} \langle A_l \otimes B_l \rangle| \leq 1 \tag{3}$$

Notations used in the above inequality are detailed below:

- $\langle A_l \otimes B_l \rangle = \text{Tr}(\rho_{AB}(A_l \otimes B_l))$
- $\rho_{AB} \in \mathbb{H}_{\mathbb{A}} \otimes \mathbb{H}_{\mathbb{B}}$ is any bipartite quantum state [47].
- $A_l = \hat{a}_l \cdot \overrightarrow{\sigma}$, $B_l = \hat{b}_l \cdot \overrightarrow{\sigma}$, $\hat{a}_l, \hat{b}_l \in \mathbb{R}^3$ denote real orthonormal vectors. $A_l B_l$ thus denote inputs of Alice and Bob.
- $\nu = \{\hat{a}_1, \hat{a}_2, \ldots . \hat{a}_n, \hat{b}_1, \hat{b}_2, \ldots, \hat{b}_n\}$ stands for the collection of measurement directions.

In case, dimension of each of local Hilbert spaces $\mathbb{H}_{\mathbb{A}}, \mathbb{H}_{\mathbb{B}}$ is 2, ρ_{AB} is given by Eq. (1). Violation of Eq. (3) guarantees both way steerability of ρ_{AB} in the sense that it is steerable from A to B and vice versa.

Steering phenomenon remaining invariant under local unitary transformations, the analytical expressions of the steering inequality remain unaltered if the simplified form [Eq. (2)] of two qubit state ρ_{AB} is considered. The analytical expression of the upper bound of corresponding inequality for 3 settings is given by [47]:

$$\begin{aligned} \text{Max}_{\nu} \mathcal{F}_3(\rho_{AB}, \nu) &= \sqrt{\mathrm{t}_{11}^2 + \mathrm{t}_{22}^2 + \mathrm{t}_{33}^2}, \\ &= \sqrt{\text{Tr}(T^t T)} \\ &= \sqrt{\text{Tr}(W^t W)} \end{aligned} \tag{4}$$

where W and T denote the correlation tensor corresponding to density matrix representation of ρ_{AB} given by Eqs. (1) and (2), respectively. Last equality in Eq. (4) holds as trace of a matrix is unitary equivalent. Hence, by the linear inequality (Eq. 3) (for $n = 3$), any two qubit state ρ_{AB}(shared between A and B) is both-way F_3 steerable if:

$$\mathcal{S}_{AB} = Tr[T_{AB}^T T_{AB}] > 1. \tag{5}$$

Equation (5) gives only a sufficient condition detecting steerability. So if any state violates Eq. (5), the state may be steerable, but its steerability remains undetected by CJWR inequality [Eq. (3) for $n = 3$]. Any state violating Eq. (5) may be referred to as F_3 unsteerable state in the sense that the state is unsteerable up to CJWR inequality for three settings.

3.3 Bell nonlocality in three settings measurement scenario

Consider a bipartite measurement scenario involving three dichotomic measurements settings (on each side). Such a scenario is often referred to as $(3, 3, 2, 2)$ measurement scenario. CHSH is not the only possible facet inequality in $(3, 3, 2, 2)$ scenario [66,67]. A complete list of facet inequalities of Bell polytope (for this measurement scenario) was computed in [67]. There exists only one Bell inequality inequivalent to CHSH inequality. That inequivalent facet inequality is referred to as the I_{3322} inequality [48]. Denoting local measurements of Alice and Bob as A_1, A_2, A_3 and B_1, B_2, B_3, respectively, and the outcomes of each of this measurement as ± 1, I_{3322} inequality takes the form [48]:

$$\begin{aligned} -2P_{B_1} - P_{B_2} - P_{A_1} + P_{A_1B_1} + P_{A_1B_2} + P_{A_1B_3} \\ +P_{A_2B_1} + P_{A_2B_2} - P_{A_2B_3} + P_{A_3B_1} - P_{A_3B_2} \leq 0, \end{aligned} \tag{6}$$

where $\forall\, i, j = 1, 2, 3$, $P_{B_i} = p(1|B_i)$, $P_{A_i} = p(1|A_i)$ denote marginal probabilities and $P_{A_iB_j} = p(11|A_iB_j)$ stands for the joint probability terms. In terms of these probability terms, CHSH inequality [3] takes the form:

$$-(P_{A_1} + P_{B_1} + P_{A_2B_2}) + P_{A_1B_1} + P_{A_1B_2} + P_{A_2B_1} \leq 0 \tag{7}$$

There exist quantum states which violate above inequality [Eq. (6)] but satisfy CHSH inequality [Eq. (7)] and vice-versa [48]. Violation of anyone of CHSH [Eq. (7)] or I_{3322} inequality [Eq.(6)] guarantees nonlocality of corresponding correlations in $(3, 3, 2, 2)$ scenario. Conversely, as these two are the only inequivalent facet inequalities of Bell-local polytope, so any correlation satisfying both Eqs. (6, 7) is Bell-local in $(3, 3, 2, 2)$ scenario.

3.4 Reduced steering

Notion of reduced steering has emerged in context of manifesting multipartite steering with the help of bipartite steering [68]. Consider an n-partite quantum state $\varrho_{1,2,\ldots,n}$ shared between n parties $A_1, A_2, \ldots, A_n$ If any one of these parties A_i(say) can steer the particle of another party say $A_j (i \neq j)$ without aid of any of the remaining parties $A_k (k \neq i, j)$, then the n-partite original state $\varrho_{1,2,\ldots,n}$ is said to exhibit reduced steering. So reduced steering is one notion of steerability of $\varrho_{1,2,\ldots,n}$. Technically speaking $\varrho_{1,2,\ldots,n}$ is steerable if at least one of the bipartite reduced states $\varrho_{i,j}$ is steerable.

4 Hidden steerability in linear network

As already mentioned before, we focus on steering activation in quantum network scenario involving qubits such that steerable correlations are generated between two distant parties who do not share any direct common past. We start with an entanglement swapping network involving three parties.

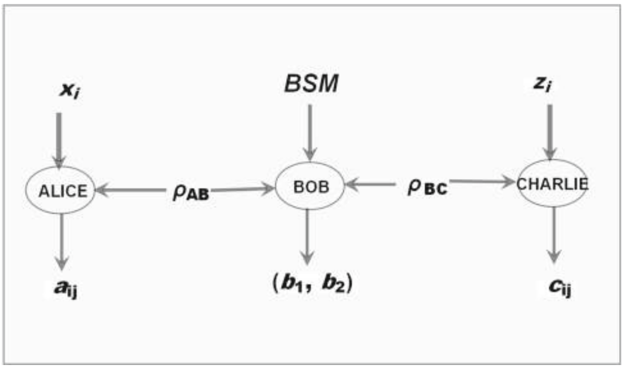

Fig. 1 A network of three parties Alice, Bob and Charlie. Alice and Bob share an entangled state ρ_{AB} and that the state shared between Bob and Charlie is ρ_{BC}. Bob performs Bell basis measurement(BSM) on his two particles and communicates the results to Alice and Charlie who then perform projective measurements on their conditional state

4.1 Linear three party network scenario

Consider a network of three parties Alice, Bob and Charlie arranged in a linear chain (see Fig. 1). Let ρ_{AB} denote the entangled state shared between Alice and Bob, whereas entangled state ρ_{BC} be shared between Bob and Charlie. So initially Alice and Charlie do not share any physical state. Let one way classical communication be allowed between parties sharing a state. To be more specific Bob can communicate to each of Alice and Charlie. Alice and Charlie are thus the two non-interacting parties.

First Bob performs joint measurement on his two qubits in the Bell basis:

$$|\phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, |\psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}. \quad (8)$$

Let $\vec{v} = (b_1 b_2)$ denote the outcome of Bob: $(0,0), (0,1), (1,0), (1,1)$ correspond to $|\phi^+\rangle$, $|\phi^-\rangle$, $|\psi^+\rangle$ and $|\psi^-\rangle$. Bob then communicates the results to Alice and Charlie. Let $\rho_{AC}^{(b_1 b_2)}$ be the conditional state shared between Alice and Charlie when Bob obtains the outcome $\vec{b} = (b_1 b_2)$. Each of Alice and Charlie now performs one of three arbitrary projective measurements on their respective qubits. Let x_i and $z_i (i = 1, 2, 3)$ denote the measurement settings of Alice and Charlie with a_{ij} and $c_{ij} (j = 0, 1)$ denoting the binary outputs corresponding to x_i and z_j, respectively. Bipartite correlations arising from the local measurements of Alice and Charlie are then used to test CJWR inequality for three settings:

$$\frac{1}{\sqrt{3}} |\langle A_1 \otimes C_1\rangle + \langle A_2 \otimes C_2\rangle + \langle A_3 \otimes C_3\rangle| \leq 1 \quad (9)$$

Such a testing of the conditional states is required to check activation of steerability in the network. Idea of steerability activation detection is detailed below.

4.2 Steering activation in network

Phenomenon of steering activation is observed if both the initial states ρ_{AB} and ρ_{BC} are F_3 unsteerable, whereas at least one of the four conditional states $\rho_{AC}^{(00)}, \rho_{AC}^{(01)}, \rho_{AC}^{(10)}, \rho_{AC}^{(11)}$ is F_3 steerable. Precisely speaking, activation occurs if both ρ_{AB} and ρ_{BC} violate Eq. (5), whereas $\rho_{AC}^{b_1 b_2}$ satisfies the same for at least one possible pair (b_1, b_2). Any pure entangled state being F_3 steerable, no activation is possible if one or both of the initial states ρ_{AB} and ρ_{BC} possess pure entanglement. So the periphery of analyzing steerability activation encompasses only mixed entangled states. We next provide with an instance of activation observed in the network.

4.3 An instance of activation

Let us now consider the following families of two qubit states:

$$\gamma_1 = (1-p)|\varphi\rangle\langle\varphi| + p|00\rangle\langle 00| \quad (10)$$
$$\gamma_2 = (1-p)|\varphi\rangle\langle\varphi| + p|11\rangle\langle 11| \quad (11)$$

where $|\varphi\rangle = \sin\alpha|01\rangle + \cos\alpha|10\rangle$, $0 \leq \alpha \leq \frac{\pi}{4}$ and $0 \leq p \leq 1$. These class of states were used for the purpose of increasing maximally entangled fraction in an entanglement swapping network [69]. Each of these families violates Eq. (5) if:

$$2((1-p)\sin 2\alpha)^2 + (2p-1)^2 \leq 1 \quad (12)$$

Now let ρ_{AB} and ρ_{BC} be any member of the family given by γ_1 and γ_2 [Eqs. (10,11)], respectively, such that the state parameters satisfy Eq. (12). When Bob's particles get projected along $|\phi^{\pm}\rangle$, each of the conditional states $\rho_{AC}^{00}, \rho_{AC}^{01}$ is steerable [satisfying Eq. (5)] if:

$$\frac{1}{N_1}(9 - 26p + 25p^2 + 4(3 - 8p + 5p^2)\cos(2\alpha) + 3(-1+p)^2\cos(4\alpha)) > 1 \quad (13)$$

where $N_1 = 2(-1-p+(-1+p)\cos(2\alpha))^2$. Similarly if Bob's output is $|\psi^{\pm}\rangle$, steerability of each of $\rho_{AC}^{10}, \rho_{AC}^{11}$ is guaranteed if

$$\frac{1}{N_2}(8(-1+p)^4\sin(2\alpha)^4 + N_3) > 1, \quad (14)$$

where $N_2 = (3 - 2p + 3p^2 - 4(-1+p)p\cos(2\alpha) + (-1+p)^2\cos(4\alpha))^2$ and $N_3 = (3 - 10p + 11p^2 + 4(-1+p)p\cos(2\alpha) + (-1+p)^2\cos(4\alpha))^2$. There exist state parameters (p, α) which satisfy Eqs. (12, 13). This in turn indicates that there exist states from the two families [Eqs. (10,11)] for which steerability is activated for Bob obtaining 00 or 01 output (see Fig. 2). For example, activation is observed for all members from these two families characterized by $\alpha = 0.1$, and $p \in (0.001, 0.331)$. However, in case conditional state

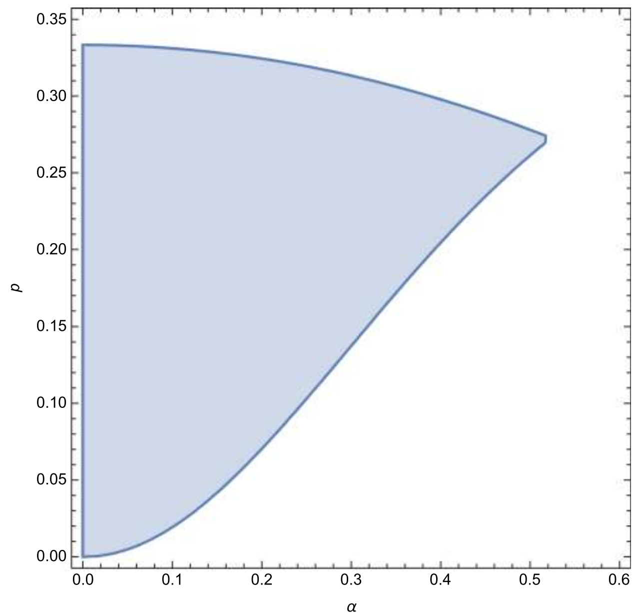

Fig. 2 Shaded region is a subspace in the parameter space (p, α) of the family of states given by Eqs. (10, 11). It indicates region of steering activation [as detected by Eq. (5)] obtained in the entanglement swapping protocol (Fig. 1) when Bob obtains either $|\phi^+\rangle$ or $|\phi^-\rangle$. It should be noted here that none of the conditional states $\rho_{AC}^{(00)}$, $\rho_{AC}^{(01)}$ is Bell nonlocal in three binary measurement settings scenario [48]

$\rho_{AC}^{(10)}$ or $\rho_{AC}^{(11)}$ is obtained, activation of steering is not observed.

To this end one may note that a conditional state satisfying anyone of Eq. (13) or Eq. (14) is Bell-local in $(3, 3, 2, 2,)$ scenario, i.e., it violates neither I_{3322} inequality [Eq. (6)] nor CHSH inequality [Eq. (7)].

4.4 Measurement settings detecting steerability

As already mentioned before, for the purpose of investigating activation, criterion [Eq. (5)] used as a sufficient criterion for detecting steerability of conditional states is a closed form of the upper bound of violation of CJWR inequality for three settings [Eq. (9)[. It may be pointed out that the two parties sharing the conditional state in the network being Alice and Charlie, in Eq. (9), observables C_i considered unlike that of B_i [used in Eq. (3)]. Now, as the closed form involves only state parameters [47], in case any state satisfies the criterion given by Eq. (5), state is steerable. But no information about measurement settings involved in detecting steerability of the state can be obtained. However, from practical view point, it is interesting to know suitable measurement settings which help in steering the states. For that, given a two qubit state, suitable measurement settings are those projective measurements (for each of the two parties) for which the state considered violates Eq. (9). $A_i = \vec{a}_i.\vec{\sigma}$ and $C_i = \vec{c}_i.\vec{\sigma} (i = 1, 2, 3)$ denote projection measurements of Alice and Charlie, respectively. As mentioned in Sect. 3, for violation of CJWR inequality [Eq. (9)], each of Alice and Charlie performs projective measurements in orthogonal directions: $\vec{a_i}.\vec{a_j} = 0 = \vec{c_i}.\vec{c_j}, \forall\, i \neq j$. CJWR inequality being symmetric [39], violation of Eq. (9) implies that the corresponding state is steerable from Alice to Charlie and also from Charlie to Alice. Now, for obvious reasons choice of appropriate settings is state specific. For providing some specific examples of suitable measurement settings, we next consider the instance of activation provided in Sect. 4.3.

Consider a particular member from each of the two families [Eqs. (10, 11)] characterized by $(p, \alpha) = (0.214, 0.267)$. None of these two states is steerable [up to Eq. (5)]. So none of these two states violate Eq. (9). Let these two states be used in the linear network. In case Bob gets output $(0, 0)$ or $(0, 1)$, conditional state ρ_{AC}^{00} or ρ_{AC}^{01}, shared between Alice and Charlie, violates Eq. (9) when Alice projects her particle in anyone of the three following orthogonal directions: $(0, 0, 1)$, $(0, -1, 0)$, $(-1, 0, 0)$ and Charlie's projective measurement directions are given by: $(0, 0, 1)$, $(0, 1, 0)$, and $(-1, 0, 0)$. It may be noted here that these are not the only possible directions for which violation of Eq. (9) is observed. Alternate measurement directions may also exist. However, there exists no measurement settings of Alice and Charlie for which the conditional states ρ_{AC}^{10} or ρ_{AC}^{11} violate Eq. (9). So steering activation is possible [up to Eq. (9)] in case Bob obtains output 00 or 01 only.

Getting instances of steering activation in the network, an obvious question arises next: can hidden steerability be observed for arbitrary two qubit states? This however turns out to be impossible in three measurement setting projective measurement scenario(for the non-interacting parties) when one uses Eq. (5) as steerability detection criterion [47]. We now analyze arbitrary two qubit states in this context.

4.5 Characterization of arbitrary two qubit states

Let two arbitrary states be initially considered in the swapping protocol. In density matrix formalism the states are represented as:

$$\rho_{AB} = \frac{1}{4}(\mathbb{I}_{2\times 2} + \vec{\mathfrak{u}_1}.\vec{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \vec{\mathfrak{v}_1}.\vec{\sigma} + \sum_{j_1, j_2=1}^{3} w_{1 j_1 j_2} \sigma_{j_1} \otimes \sigma_{j_2}), \quad (15)$$

$$\rho_{BC} = \frac{1}{4}(\mathbb{I}_{2\times 2} + \vec{\mathfrak{u}_2}.\vec{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \vec{\mathfrak{v}_2}.\vec{\sigma} + \sum_{j_1, j_2=1}^{3} w_{2 j_1 j_2} \sigma_{j_1} \otimes \sigma_{j_2}), \quad (16)$$

Steerability of states remains unhindered under local unitary operations. Let suitable local unitary operations be applied to the initial states for diagonalizing the correlation tensors:

$$\rho'_{AB} = \frac{1}{4}(\mathbb{I}_{2\times 2} + \vec{\mathfrak{a}_1}.\vec{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \vec{\mathfrak{b}_1}.\vec{\sigma} + \sum_{j=1}^{3} \mathfrak{t}_{1jj}\sigma_j \otimes \sigma_j), \quad (17)$$

$$\rho'_{BC} = \frac{1}{4}(\mathbb{I}_{2\times 2} + \vec{\mathfrak{a}_2}.\vec{\sigma} \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \vec{\mathfrak{b}_2}.\vec{\sigma} + \sum_{j=1}^{3} \mathfrak{t}_{2jj}\sigma_j \otimes \sigma_j), \quad (18)$$

Let both ρ'_{AB} and ρ'_{BC} be F_3 unsteerable, i.e., let both of them violate Eq. (5). Hence $\sum_{j=1}^{3}\sqrt{\mathfrak{t}^2_{1jj}}\leq 1$,$\sqrt{\mathfrak{t}^2_{2jj}}\leq 1$. We next characterize ρ'_{AB} and ρ'_{BC} by analyzing nature of the conditional states $\rho^{b_1b_2}_{AC}$. In this context, we provide three results each of which can be considered as a condition for no steering activation in the network. To be precise, if Bloch parameters of any initial two qubit states satisfy assumptions (see Table 1 for more details) of any of these three results then there will be no activation of F_3 steerability. Of these three results, two are proved analytically, whereas the last one is a numerical observation only. First, we give the two analytic results in form of two theorems.

Theorem 1 *If one or both the initial states [Eqs. (17, 18)] do not have any non-null local Bloch vector (see Table 1) then none of the conditional states* $\rho^{b_1b_2}_{AC}$ *satisfies Eq. (5).*

Proof See Appendix.A Up to the steering criterion given by Eq. (5), above result implies impossibility of steering activation in swapping network involving two qubit states whose local Bloch vectors(corresponding to both the parties) vanish under suitable local unitary operations. Maximally mixed marginals class of two qubit states has no local Bloch vector. So, activation is not possible in network involving any member from this class. So hidden steerability cannot be exploited in absence of local Bloch vectors corresponding to both the parties of a bipartite quantum state. But can the same be generated if both ρ'_{AB} and ρ'_{BC} has one non-null local Bloch vector? Following theorem provides a negative observation. □

Theorem 2 *If both the initial states* ρ'_{AB} *and* ρ'_{BC} *have only one non-null local Bloch vector, i.e.,* $\vec{\mathfrak{a}_1} = \vec{\mathfrak{a}_2} = \Theta$ *or* $\vec{\mathfrak{b}_1} = \vec{\mathfrak{b}_2} = \Theta$*(*$\Theta$ *denote null vector) then none of the conditional states* $\rho^{b_1b_2}_{AC}$ *satisfies Eq. (5).*

Proof of Theorem 2 This proof is exactly the same as that for Theorem 1 owing to the fact that here also $\sqrt{\mathrm{Tr}(\mathcal{V}^T_{b_1b_2}\mathcal{V}_{b_1b_2})} = \sqrt{\sum_{k=1}^{3}(t_{1kk}t_{2kk})^2}$ where $\mathcal{V}_{b_1b_2}$ denote correlation tensor of resulting conditional states $\rho^{(b_1b_2)}_{AC}$.

Table 1 Assumptions of three results (analyzed above) are enlisted here. The correlation tensor of each of the two initial states ρ'_{AB} and ρ'_{BC} remain arbitrary. Restrictions are imposed over the local Bloch parameters only

Result	Assumptions	Steerability Activation
Theorem.1	$(\vec{\mathfrak{a}_i}, \vec{\mathfrak{b}_i}) = (\Theta, \Theta)\,\forall\, i$ or $(\vec{\mathfrak{a}_i}, \vec{\mathfrak{b}_i}) = (\Theta, \Theta)$ for $i = 1$ or $(\vec{\mathfrak{a}_i}, \vec{\mathfrak{b}_i}) = (\Theta, \Theta)$ for $i = 2$	No
Theorem.2	$\vec{\mathfrak{a}_1} = \vec{\mathfrak{a}_2} = \Theta$ or $\vec{\mathfrak{b}_1} = \vec{\mathfrak{b}_2} = \Theta$	No
Numerical Observation	$\vec{\mathfrak{a}_1} = \vec{\mathfrak{b}_2} = \Theta$ or $\vec{\mathfrak{b}_1} = \vec{\mathfrak{a}_2} = \Theta$	No

Note that in Theorem 2, $\vec{\mathfrak{a}_1} = \vec{\mathfrak{a}_2} = \Theta$ or $\vec{\mathfrak{b}_1} = \vec{\mathfrak{b}_2} = \Theta$ is considered. But what if $\vec{\mathfrak{a}_1} = \vec{\mathfrak{b}_2} = \Theta$ or $\vec{\mathfrak{b}_1} = \vec{\mathfrak{a}_2} = \Theta$? Does activation occurs in such case? Numerical evidence suggests a negative response to this query:
Numerical Observation: *If* $\vec{\mathfrak{a}_1} = \vec{\mathfrak{b}_2} = \Theta$ *or* $\vec{\mathfrak{b}_1} = \vec{\mathfrak{a}_2} = \Theta$ *then none of the conditional states* $\rho^{b_1b_2}_{AC}$ *satisfies Eq. (5).*

Justification of this observation is based on the fact that numerical maximization of the steerability expression [Eq. (5)] corresponding to each possible conditional state $\rho^{b_1b_2}_{AC}$ gives 1 under the constraints that both the initial quantum states(ρ'_{AB}, ρ'_{BC}). Consequently, none of the conditional states satisfies Eq. (5) if none of ρ'_{AB}, ρ'_{BC} satisfies Eq. (5).

Above analysis points out the fact that for revealing hidden steerability, each of ρ'_{AB} and ρ'_{BC} should have non-null local Bloch vectors corresponding to both the parties. However that condition is also not sufficient for activation. In case, correlation tensor of any one of them is a null matrix, the state is a separable state. When such a state is considered as an initial state in the network, none of the conditional states is entangled and thereby activation of steerability becomes impossible. So, when steerability is activated in the network following are the necessary requirements:

- All of the local Bloch vectors must be non-null: $\vec{\mathfrak{a}_i} \neq \Theta$,$\vec{\mathfrak{b}_i} \neq \Theta\,\forall\, i$ and
- Both the initial states should have non-null correlation tensors.

However, the above conditions are only necessary for activation purpose but are not sufficient for the same. We next provide illustration with specific examples in support of our claim. □

4.5.1 Illustration

Let us now analyze the classes of states given by Eqs. (10, 11) in perspective of above characterization. Both the families of initial states [Eqs. (10, 11)] have local Bloch vectors: $\vec{\mathfrak{a}_1} = (0, 0, p - \cos(2\alpha)(1 - p))$, $\vec{\mathfrak{b}_1} = (0, 0, p + \cos(2\alpha)(1 - p))$, $\vec{\mathfrak{a}_2} = (0, 0, -p - \cos(2\alpha)(1 - p))$, $\vec{\mathfrak{b}_2} = (0, 0, -p + \cos(2\alpha)(1 - p))$. Local Bloch vectors are non-null for $\cos(2\alpha) \neq \pm\frac{p}{1-p}$. Correlation tensors of the states from both the families are given by $\text{diag}((1-p)\sin(2\alpha), (1-p)\sin(2\alpha), 2p-1)$. Clearly activation is not observed for all family members having non-null local Blochs as well as non-null correlation tensors. For instance, consider $(p, \alpha) = (0.6, 0.6)$. Bloch parameters of corresponding states are given by:

- $\vec{\mathfrak{a}_1} = (0, 0, 0.455057))$, $\vec{\mathfrak{b}_1} = (0, 0, 0.744943)$,
- $\vec{\mathfrak{a}_2} = (0, 0, -0.455057))$, $\vec{\mathfrak{b}_2} = (0, 0, -0.744943)$,
- $\text{diag}(t_{i11}, t_{i22}, t_{i33}) = \text{diag}(0.372816, 0.372816, 0.2)$, $\forall\, i$

No steering activation is observed when these two states are used in the network. This in turn implies that the criteria given in 4.5 are only necessary but not sufficient to ensure activation in the network. Now, as already discussed in Sect. 4.3, there exist members from these families (see Fig. 2) which when used in the swapping network steering activation is observed.

Network scenario considered so far involved two states shared between three parties. However, will increasing length of the chain, hence increasing number of initial states be useful for the purpose of revealing hidden steerability? Though general response to this query is non-trivial, we consider a star network configuration of four parties to give instances of activation of reduced steering.

5 Nonlinear swapping network involving $n \geq 3$ states

Consider $n+1 (n \geq 3)$ number of parties $A_1, A_2, \ldots, A_n$ and B. Let n bipartite states $\varrho_i (i = 1, 2, \ldots, n)$ be shared between the parties such that ϱ_i is shared between parties B and $A_i (i = 1, 2, \ldots, n)$ (see Fig.3). B performs a joint measurement on his share of qubits from each ϱ_i and communicates outputs to the other parties $A_i (i = 1, 2, \ldots, n)$. Reduced steering of each of the conditional n-partite states is checked. To be precise, it is checked whether at least one possible bipartite reduced state of at least one of the conditional states satisfies Eq. (5). In case at least one of the conditional states has reduced steering when none of $\varrho_i (i = 1.2 \ldots, n)$ satisfies Eq. (5), activation of steerability is obtained. Activation is thus observed when one of the n parties sharing n-partite conditional state can steer the particles of another party without any assistance from remaining $n - 2$ parties sharing the same.

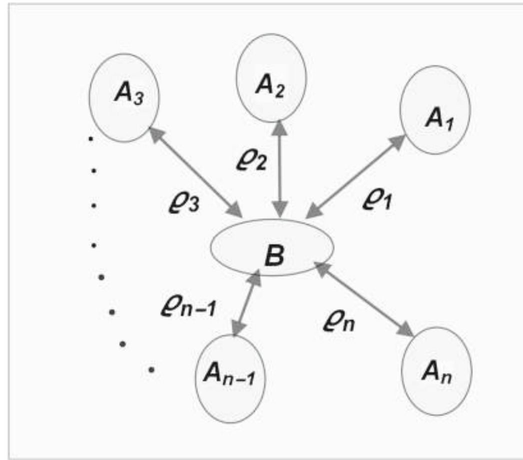

Fig. 3 Schematic Diagram of a star network. For $i = 1, 2, \ldots, n$, bipartite state ϱ_i is shared between parties B and A_i. Party B performs joint measurement on state of his n particles and communicates his output to each of $A_1, A_2, \ldots, A_n$. Reduced steering of corresponding conditional state shared between $A_1, A_2, \ldots, A_n$ is checked

Consider a specific instance of $n = 3$. Let each of $\varrho_1, \varrho_2, \varrho_3$ be a member of the family of states given by Eq.(10) with $p = p_1, p_2, p_3$ for $\varrho_1, \varrho_2, \varrho_3$, respectively. Let B perform joint measurement in the following orthonormal basis:

$$
\begin{aligned}
|\delta_1\rangle &= \frac{1}{\sqrt{3}}(|001\rangle + |100\rangle + |010\rangle) \\
|\delta_2\rangle &= \frac{1}{\sqrt{3}}(|010\rangle - |100\rangle + |000\rangle) \\
|\delta_3\rangle &= \frac{1}{\sqrt{3}}(-|010\rangle + |001\rangle + |000\rangle) \\
|\delta_4\rangle &= \frac{1}{\sqrt{3}}(|100\rangle + |000\rangle - |001\rangle) \\
|\delta_5\rangle &= \frac{1}{\sqrt{3}}(|101\rangle + |110\rangle + |011\rangle) \\
|\delta_6\rangle &= \frac{1}{\sqrt{3}}(|110\rangle - |101\rangle + |111\rangle) \\
|\delta_7\rangle &= \frac{1}{\sqrt{3}}(-|110\rangle + |111\rangle + |011\rangle) \\
|\delta_8\rangle &= \frac{1}{\sqrt{3}}(|111\rangle + |101\rangle - |011\rangle) \qquad (19)
\end{aligned}
$$

When B's particles get projected along δ_j, let $\rho^{(j)} (j = 1, \ldots, 8)$ denote the conditional state shared between A_1, A_2, A_3. Reduced steering of each of the conditional states is checked in terms of the steering inequality given by Eq. (5). Now, let all three initial states $\varrho_1, \varrho_2, \varrho_3$ violate Eq. (5). When B's state gets projected along any one of $\delta_1, \delta_6, \delta_7, \delta_8$ [Eq. (19)], for some state parameters (p, α), each of corresponding condi-

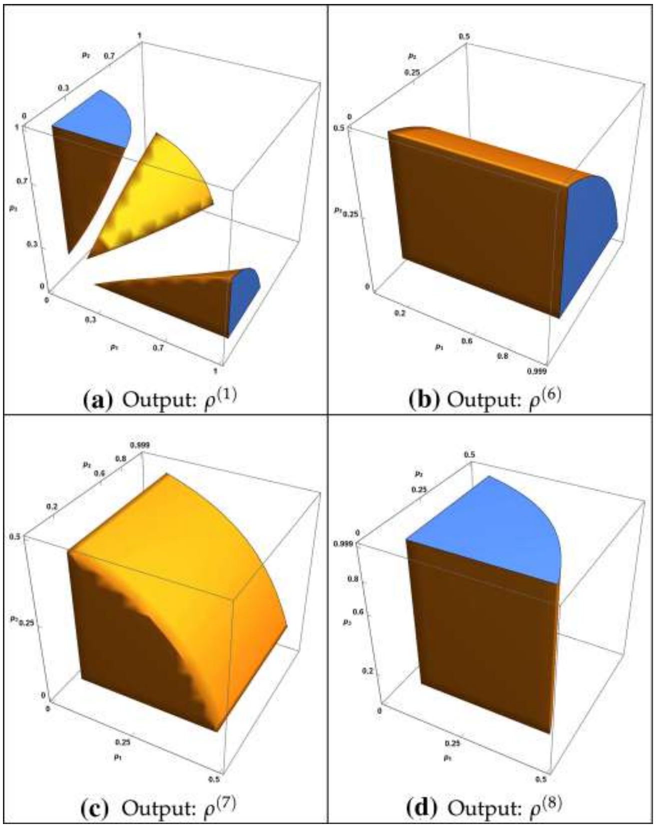

Fig. 4 Shaded region in each of four sub figures in the grid gives the steering activation region obtained stochastically depending on the different possible outputs of party B's measurement in orthonormal basis [Eq. (19)]. Here, the star network scenario (Fig. 3) involves three non-identical states from the class given by Eq. (10) for $\alpha = 0.2$. Starting from the top row and moving from left to right, shaded regions indicates reduced steering activation when B's particles get projected along $\delta_1, \delta_6, \delta_7$ and δ_8, respectively

tional states has reduced steering. Region of activation is thus observed (see Fig. 4). Some particular instances of activation are enlisted in Table 2. At this point it should be pointed out that none of the reduced states corresponding to the conditional states violates neither I_{3322} inequality [Eq. (6)] nor CHSH inequality [Eq. (7)] and hence are Bell local (in $(3,3,2,2)$ scenario).

6 Genuine activation of steerability

Most of the research works in the field of activation scenarios analyze activation of nonclassicality of quantum states with respect to any specific detection criterion of the nonclassical feature considered. To be precise, let $\mathcal{C}$ denote a detection criterion for a specific notion of quantum nonclassicality. Activation is said to be observed in any protocol if using one or more quantum states(or identical copies of the same state), none of which satisfies $\mathcal{C}$, another quantum state is gener-

Table 2 Some specific values of state parameters are enlisted here for which stochastic steering activation(in terms of reduced steering) is observed in nonlinear network (Fig. 3)

State	p_1	p_2	Range of p_3
$\rho^{(1)}$	0.08	0.075	$(0.2, 1]$
$\rho^{(6)}$	0.08	0.075	$(0.071, 0.467]$
$\rho^{(7)}$	0.08	0.075	$(0, 071, 0.465]$
$\rho^{(8)}$	0.08	0.075	$(0.2, 1)$

To be more precise, for $\alpha = 0.2$, other parameters p_1, p_2, p_3 are specified for the three non-identical states from the class given by Eq. (10). First column in the table gives the conditional state corresponding to which activation is observed

ated(at the end of the protocol) that satisfies $\mathcal{C}$. Using detection criterion of $\mathcal{F}_3$ steerability [39,47], so far we have obtained various cases of steering activation in both linear and nonlinear quantum networks. But quite obviously such a trend of activation analysis is criterion specific and in general can be referred to as *activation of $\mathcal{F}_3$ steerability.* But here we approach to explore activation beyond the periphery of criterion specification. We refer to such activation as *genuine activation of steerability.*

Let us consider the linear chain of three parties (Fig. 1). For genuine activation we use states which satisfy some criterion of unsteerability and then explore $\mathcal{F}_3$ steerability of the conditional states resulting due to Bell basis measurement(BSM) by the intermediate party(Bob) in the protocol. Genuine activation of steerability occurs in case at least one of $\rho_{AC}^{b_1b_2}$ satisfies Eq. (5). In [70], the authors proposed an asymmetric sufficient criterion of bipartite unsteerability.

Let ρ_{AB} be any two qubit state shared between Alice and Bob(say). In density matrix formalism ρ_{AB} is then provided by Eq. (1). Consider a positive, invertible linear map Λ, whose action on ρ_{AB} is given by [70]:

$$\mathbb{I}_2 \otimes \Lambda(\rho_{AB}) = \mathbb{I}_2 \otimes \rho_B^{-1} \rho_{AB} \mathbb{I}_2 \otimes \rho_B^{-1}, \qquad (20)$$

where $\mathbb{I}_2$ is 2×2 identity matrix in Hilbert space associated with 1^{st} party and $\rho_B = \mathrm{Tr}_A(\rho_{AB})$. Let $\rho_{AB}^{(1)}$ denote the state density matrix obtained after applying the above map to ρ_{AB}. Local Bloch vector corresponding to 2^{nd} party(Bob) of $\rho_{AB}^{(1)}$ becomes a null vector [70]:

$$\rho_{AB}^{(1)} = \frac{1}{4}(\mathbb{I}_{2\times 2} + \vec{u'}.\vec{\sigma} \otimes \mathbb{I}_2 + \sum_{j_1, j_2=1}^{3} w'_{j_1 j_2} \sigma_{j_1} \otimes \sigma_{j_2}), \qquad (21)$$

On further application of local unitary operations to diagonalize correlation tensor, ρ'_{AB} ultimately becomes:

$$\rho^{(2)}_{AB} = \frac{1}{4}(\mathbb{I}_{2\times 2} + \vec{\mathfrak{u}''}.\vec{\sigma} \otimes \mathbb{I}_2 + \sum_{j=1}^{3} w''_{jj}\sigma_j \otimes \sigma_j), \tag{22}$$

$\rho^{(2)}_{AB}$ [Eq. (22)] is referred to as the *canonical form* of ρ_{AB} in [70] where the authors argued that ρ_{AB} will be unsteerable if and only if $\rho^{(2)}_{AB}$ is unsteerable. They showed that ρ_{AB} is unsteerable from Alice to Bob if [70]:

$$\text{Max}_{\hat{x}}((\vec{\mathfrak{a}}.\hat{x})^2 + 2||\mathcal{W}''\hat{x}||) \leq 1 \tag{23}$$

where $\hat{x}$ is any unit vector indicating measurement direction, $\mathcal{W}''$ denotes the correlation tensor of ρ''_{AB} and $||.||$ denotes Euclidean norm.

For our purpose we consider the unsteerability criterion given by Eq. (23). Below, we characterize arbitrary two qubit states in ambit of genuine activation of steerability.

6.1 Characterizing two qubit states

Let ρ_{AB} and ρ_{BC} [Eqs. (15, 16)] denote two arbitrary two qubit states used in the network. It turns out that local Bloch vector corresponding to first party of the initial states play a significant role in determining possibility of genuine activation of steering in the network. Next we give two results. While one of those is provided with an analytical proof, analysis of the other one relies on numerical optimization. We first state the analytical result.

Theorem 3 *If canonical forms of both the initial states ρ_{AB} and ρ_{BC} [Eqs. (15, 16)] satisfy the unsteerability criterion [Eq. (23)], then genuine activation of steerability is impossible if both of them have null local Bloch vector corresponding to first party, i.e., $\vec{u_1}, \vec{u_2} = \Theta$.*

Proof See appendix. Genuine activation being impossible in case both $\vec{u_1}, \vec{u_2}$ are null vectors, an obvious question arises whether it is possible in case at least one of $\vec{u_1}, \vec{u_2} \neq \Theta$. We provide next result in this context. As numerical procedure is involved in corresponding calculations(see Appendix C), our next result will be considered as a numerical observation only.

Numerical Observation: *If canonical forms of both ρ_{AB} and ρ_{BC}* [Eqs. (15, 16)] *satisfy the unsteerability criterion* [Eq. (23)], *then genuine activation of steerability is impossible if any one of ρ_{AB} or ρ_{BC} has null local Bloch vector corresponding to first party, i.e., at least one of $\vec{u_1}, \vec{u_2} = \Theta$.* Justification of this observation is given in Appendix C Clearly, the above two results, combined together provide a necessary criterion for genuine activation of steerability: *When canonical forms of both the initial states satisfy Eq. (23), if steering is genuinely activated in the network then both the initial states must have non-null local Bloch vectors corresponding to first party, i.e., $\vec{u_1} \neq \Theta$, $\vec{u_2} \neq \Theta$.* We next provide with examples in this context. □

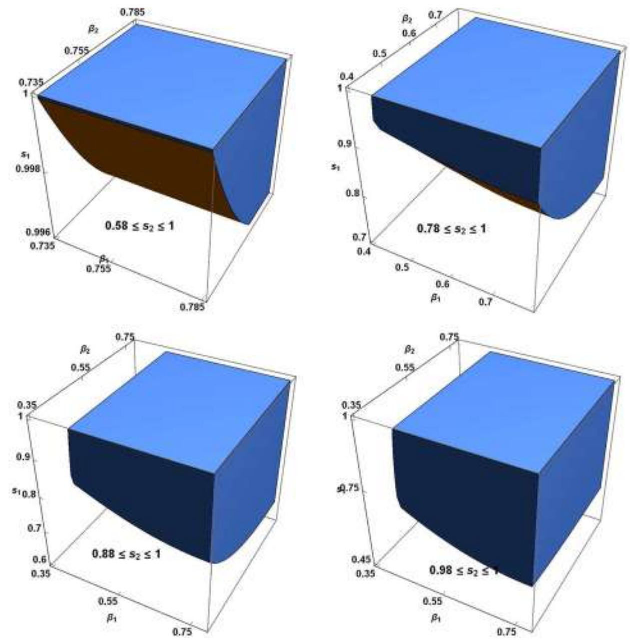

Fig. 5 Shaded regions in the sub figures give region of genuine activation of steerability for different ranges of state parameter s_2. None of the steerable conditional states obtained in the protocol is Bell nonlocal in $(3, 3, 2, 2)$ measurement scenario

6.2 Examples

Consider a family of states [70]:

$$\Omega = s|\chi\rangle\langle\chi| + (1-s)\Omega^1 \otimes \frac{\mathbb{I}_2}{2}, \tag{24}$$

where $|\chi\rangle = \cos(\beta)|00\rangle + \sin(\beta)|11\rangle$, $0 \leq s \leq 1$, $\mathbb{I}_2$ is $2x2$ identity matrix in Hilbert space associated with 2^{nd} party and Ω^1 is the reduced state of first party obtained by tracing out second party from $|\chi\rangle\langle\chi|$, i.e., $\Omega^1 = \cos^2(\beta)|0\rangle\langle 0| + \sin^2(\beta)|1\rangle\langle 1|$. For $\beta \neq \frac{\pi}{4}$, any member from this class has non-null local Bloch vector corresponding to first party:$(0, 0, \cos(2\beta))$. Canonical form [Eq. (22)] of any member of this class satisfies Eq. (23) if [70]:

$$\cos^2(2\beta) \geq \frac{2s-1}{(2-s)s^3}. \tag{25}$$

Let two non-identical members Ω_1 and Ω_2 from this class [Eq. (24)] be used in the entanglement swapping protocol (Fig. 1). Let (β_1, s_1) and (β_2, s_2) be state parameters of Ω_1 and Ω_2, respectively. Let both Ω_1 and Ω_2 be unsteerable. Now, for some values of the state parameters, the conditional states generated in the protocol turn out to be steerable (see Fig. 5) as they satisfy Eq. (5). Range of parameter s_2(for some fixed value of other three parameters (β_1, β_2, s_1)) for which genuine activation occurs, is provided in Table 3.

Now, as discussed above, the criterion of both the initial unsteerable states having non-null local Bloch vector(corresponding to first party) is necessary for

Table 3 For some specific values of parameters (β_1, β_2, s_1), of Ω_1, Ω_2 range of steerability activation other parameter s_2 is specified

State	β_1	β_2	s_1	Range of s_2
ρ_{AC}^{00}	0.75	0.76	0.99	[0.58, 1]
ρ_{AC}^{01}	0.65	0.6	0.97	[0.78, 1]
ρ_{AC}^{10}	0.55	0.55	0.9	[0.88, 1]
ρ_{AC}^{11}	0.6	0.55	0.8	[0.98, 1]

First column specifies the conditional state corresponding to which activation is observed

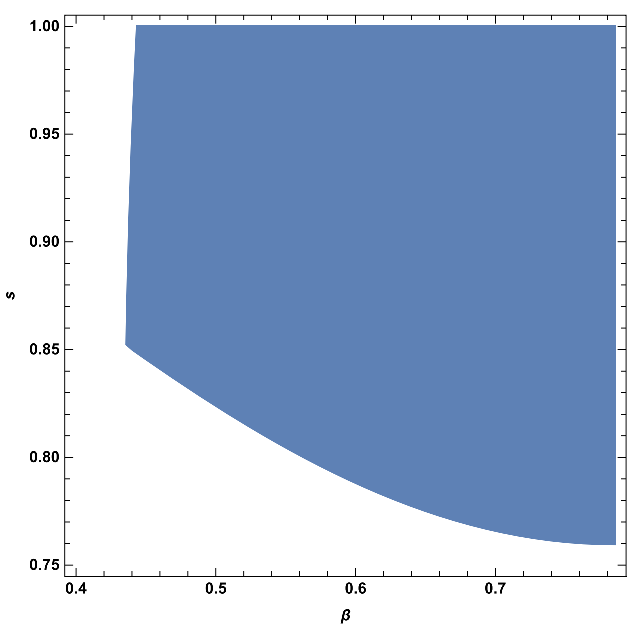

Fig. 6 Genuine activation region obtained for any possible conditional state when two identical copies of a state from Ω class are used in the network

genuine activation. The criterion however turns out to be insufficient for the same. We next provide an example in support of our claim. Consider two distinct members Ω_3, Ω_4 from the family of states given by Eq. (24) corresponding to the parameters: $(\beta_3, s_3) = (0.1, 0.7)$ and $(\beta_4, s_4) = (0.3, 0.59)$. Local Bloch vectors(corresponding to first party) of Ω_3 and Ω_4 are $(0, 0, 0.980067)$ and $(0, 0, 0.825336)$, respectively. Both of Ω_3 and Ω_4 satisfy the unsteerability criterion given by Eq. (25). These two states(in their canonical forms) are now used in the tripartite linear network. Bloch matrix representations of each of the conditional states are enlisted in Table 6(see Appendix D). Unsteerability criterion [Eq. (23)] is then tested for each of these conditional states. The optimal value(obtained numerically) in the maximization problem involved in Eq. (23) turns out to be less than unity for each of the conditional states (Table 6). Hence, all the conditional states are unsteerable. Consequently no genuine activation of steering is observed in the network using Ω_3 and Ω_4.

It may be noted that genuine activation occurs for any possible output of Bob when two identical copies of same state from this class are used in the network (see Fig.6). For instance, when two identical copies of Ω_1 for $\beta_1 = 0.7$ are considered as initial states, steerability is activated genuinely for $s_1 \in (0.77, 1]$.

7 Discussions

In different information processing tasks, involving steerable correlations, better efficiency of the related protocols(compared to their classical counterparts) basically rely upon quantum entanglement. Though pure entanglement is the most suitable candidate, but owing to environmental effects, mixed entanglement is used in practical scenarios. In this context, any steerable mixed entangled state is considered to be useful. In case it fails to generate steerable correlations, it will be interesting to exploit its steerability(if any) by subjecting to suitable sequence of measurements. Entanglement swapping protocol turns out to be an useful tool in this perspective. Let us consider the two families of states given by Eqs. (10, 11). Both of them are noisy versions of pure entangled states φ. To be more specific, these families are obtained via amplitude damping of φ [71]. As already discussed above, steering activation is obtained via entanglement swapping protocol for some members from these two families. This in turn point out that entanglement swapping protocol is useful in exploiting steerability from unsteerable [up to the steering criteria given by Eq. (5)] members from these two families. All such discussions in turn point out the utility of steerability activation in network scenarios from practical viewpoint. Characterization of arbitrary two qubits states will thus be helpful in exploiting utility of any given two qubit state in the ambit of steering activation [up to Eq. (5)]. That steerability of depolarized noisy versions of pure entangled states cannot be activated (in approach considered here) is a direct consequence of such characterization owing to the fact that this class of noisy states has no local Bloch vector. Apart from revealing hidden steerability, it will be interesting to explore whether the activation protocols can be implemented in any information processing task involving network scenario so as to render better results.

In [72], the authors have shown that if a two qubit state is $\mathcal{F}_3$ steerable, i.e., satisfies Eq. (5), then it is useful for teleportation. This in turn points out the utility of the activation networks discussed here in perspective of information theoretic tasks. To be more precise, consider, for example, the tripartite linear network (Fig. 1). Both the initial states ρ_{AB}, ρ_{BC} used in the network violates Eq. (5). So $\rho_{AB}(\rho_{BC})$ cannot be used to teleport qubit from Alice to Bob (from Bob to Charlie). Now, if steerability is activated in the network stochastically, resulting conditional state can be used for the purpose of teleportation. In case, activation occurs for all possible outputs of Bob, any of the four conditional states turns out to be useful in teleportation protocol.

Now our analysis of activation in network scenarios is criterion specific and we have just provided partial characterization of two qubit state space in context of genuine activation of steerability. The unsteerability criterion [70] involves maximization over arbitrary measurement directions [Eq. (23)]. Deriving closed form of this criterion, genuine activation of steerability can be analyzed further. In star network scenario choice of the specific orthonormal basis [Eq. (19)] for joint measurement by the central party(B) served our purpose to show that increasing number of states nonlinearly can yield better results compared to the standard three party network scenario (Fig. 1). Also such activation scenario is significant as hidden steerability is revealed when at least one of the n parties sharing n-partite conditional state can steer the particles of another party without co-operation from remaining $n-2$ parties. Further analysis of such form of steerability activation(via notion of reduced steering) using more general measurement settings of the central party B will be a potential direction of future research. It will also be interesting to analyze a scheme of m copies of bipartite states arranged in a linear chain where activation occurs only after projection on any of $n < m$ copies.

In [73], the authors introduced notion of network steering and network local hidden state (NLHS) models in networks involving independent sources. They have provided with no-go results for network steering in a large class of network scenarios, by explicitly constructing NLHS models. In course of their analysis they have given an instance of both way steering activation using family of Doubly-Erased Werner (DEW) states [73]. Activation phenomenon considered there did not rely on testing any detection criterion in form of steering inequality. So from that perspective, the activation example [73] is comparable with that of genuine steering activation in our work. Characterization of two qubit state space based on genuine activation of steering discussed in Sect. 6.1 thus encompasses a broader class of steering activation results compared to a specific example of activation [73]. To this end one may note that for analysis made there, authors considered not only unsteerability but also separability of the states distributed by the sources. Following that approach, incorporating entanglement content of initial unsteerable states to explore genuine activation of steering will be an interesting direction of future research.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors comment: The manuscript has no data associated with it].

Appendix A

Proof of Theorem 1 Both $\rho^{'}_{AB}$ [Eq. (17)] and $\rho^{'}_{BC}$ [Eq.(18)] violate Eq. (5). Hence, $\sum_{j=1}^{3}\sqrt{\mathfrak{t}^2_{1jj}},\sqrt{\mathfrak{t}^2_{2jj}} \leq 1$ which imply that $|\mathfrak{t}_{kjj}| \leq 1, \forall k = 1,2$ and $j = 1,2,3$. Let $\mathcal{V}_{b_1b_2}$ denote the correlation tensor of conditional state $\rho_{AC}^{(b_1b_2)}$. Now, two cases are considered: either one or both the parties have no non-null local Bloch vectors. In both the cases, $\mathrm{Tr}(\mathcal{V}^T_{b_1b_2}\mathcal{V}_{b_1b_2}) = \sum_{k=1}^{3}(t_{1kk}t_{2kk})^2$, $\forall b_1, b_2 = 0,1$. Hence, for each of $\mathcal{V}_{b_1b_2}$, $\sqrt{\mathrm{Tr}(\mathcal{V}^T_{b_1b_2}\mathcal{V}_{b_1b_2})}$ takes the form:

$$\begin{aligned}\sqrt{\mathrm{Tr}(\mathcal{V}^T_{b_1b_2}\mathcal{V}_{b_1b_2})} &= \sqrt{\sum_{k=1}^{3}(t_{1kk}t_{2kk})^2}\\ &\leq \sqrt{\sqrt{\sum_{k=1}^{3}t^4_{1kk}}\cdot\sqrt{\sum_{k=1}^{3}t^4_{2kk}}}\\ &\leq \sqrt{\sqrt{\sum_{k=1}^{3}t^2_{1kk}}\cdot\sqrt{\sum_{k=1}^{3}t^2_{2kk}}}\\ &\leq 1. \end{aligned} \tag{26}$$

The second inequality holds as $|\mathfrak{t}_{kjj}| \leq 1, \forall k = 1,2$ and $j = 1,2,3$ and the last is due to the fact that none of the initial states satisfies Eq. (5). □

Appendix B

Proof of Theorem 3 Here, $\vec{u_1} = \vec{u_2} = \Theta$. ρ_{AB} and ρ_{BC} thus have the form:

$$\rho_{AB} = \frac{1}{4}(\mathbb{I}_{2\times2} + \mathbb{I}_2 \otimes \vec{\mathfrak{v}_1}.\vec{\sigma} + \sum_{j_1,j_2=1}^{3} w_{1j_1j_2}\sigma_{j_1}\otimes\sigma_{j_2}),$$

$$\rho_{BC} = \frac{1}{4}(\mathbb{I}_{2\times2} + \mathbb{I}_2 \otimes \vec{\mathfrak{v}_2}.\vec{\sigma} + \sum_{j_1,j_2=1}^{3} w_{2j_1j_2}\sigma_{j_1}\otimes\sigma_{j_2}),$$

Let Λ [Eq. (20)] be applied on both ρ_{AB} and ρ_{BC} followed by local unitary operations(to diagonalize the correlation tensors). Let $\rho^{(2)}_{AB}$ and $\rho^{(2)}_{BC}$ denote the respective canonical forms [Eq. (22)] of ρ_{AB} and ρ_{BC} [70]:

$$\rho^{(2)}_{AB} = \frac{1}{4}(\mathbb{I}_{2\times2} + \sum_{j=1}^{3} w^{''}_{1jj}\sigma_j\otimes\sigma_j), \tag{27}$$

$$\rho^{(2)}_{BC} = \frac{1}{4}(\mathbb{I}_{2\times2} + \sum_{j=1}^{3} w^{''}_{2jj}\sigma_j\otimes\sigma_j), \tag{28}$$

Now $\rho^{(2)}_{AB}$ and $\rho^{(2)}_{BC}$ both satisfy unsteerability criterion given by Eq. (23). This in turn gives:

$$\mathrm{Max}_{x_1,x_2,x_3}\sqrt{\sum_{j=1}^{3}(x_j w^{''}_{kjj})^2} \leq \frac{1}{2}, \quad k = 1,2 \tag{29}$$

where $\hat{x} = (x_1, x_2, x_3)$ denotes a unit vector. □

We next perform maximization over $\hat{x}$ so as to obtain a closed form of the unsteerability criterion in terms of elements of correlation tensors of the initial states $\rho^{(2)}_{AB}$ and $\rho^{(2)}_{BC}$. *Maximization over unit vector* $\hat{x}$: Taking $\hat{x} = (\sin(\theta)\cos(\phi), \sin(\theta)\sin(\phi), \cos(\theta))$, maximization problem in L.H.S. of Eq. (41) can be posed as:

$$\text{Max}_{\theta,\phi}\sqrt{A(\theta,\phi)} \tag{30}$$

where,

$$A(\theta,\phi) = \sin^2(\theta)(\cos^2(\phi)(w^{''}_{k11})^2 + \sin^2(\phi)(w^{''}_{k22})^2) + \cos^2(\theta)(w^{''}_{k33})^2 \tag{31}$$

Now for any $g_1, g_2 \geq 0$, $\text{Max}_\kappa(g_1\cos^2(\kappa) + g_2\sin^2(\kappa))$ is g_1 if $g_1 > g_2$ and g_2 when $g_2 > g_1$. This relation is used for maximizing $A(\theta,\phi)$. In order to consider all possible values of $(w^{''}_{k11})^2$, $(w^{''}_{k22})^2$ and $(w^{''}_{k33})^2$, we consider the following cases:

Case1:$(w^{''}_{k11})^2 > (w^{''}_{k22})^2$: Then $\text{Max}_\phi A(\theta,\phi)$ gives:

$$B(\theta) = \sin^2(\theta)(w^{''}_{k11})^2 + \cos^2(\theta)(w^{''}_{k33})^2 \tag{32}$$

Subcase1: $(w^{''}_{k11})^2 > (w^{''}_{k33})^2$, i.e., $(w^{''}_{k11})^2 = \text{Max}_{j=1,2,3}(w^{''}_{kjj})^2$: Then $\text{Max}_\theta B(\theta) = (w^{''}_{k11})^2$. Hence,

$$\text{Max}_{\theta,\phi}\sqrt{A(\theta,\phi)} = |w^{''}_{k11}|. \tag{33}$$

Subcase2:$(w^{''}_{k11})^2 < (w^{''}_{k33})^2$, i.e., $(w^{''}_{k22})^2 < (w^{''}_{k11})^2 < (w^{''}_{k33})^2$: Then $\text{Max}_\theta B(\theta) = (w^{''}_{k33})^2$. Hence,

$$\text{Max}_{\theta,\phi}\sqrt{A(\theta,\phi)} = |w^{''}_{k33}|. \tag{34}$$

Case2:$(w^{''}_{k11})^2 < (w^{''}_{k22})^2$: Then $\text{Max}_\phi A(\theta,\phi)$ gives:

$$B(\theta) = \sin^2(\theta)(w^{''}_{k22})^2 + \cos^2(\theta)(w^{''}_{k33})^2 \tag{35}$$

Subcase1:$(w^{''}_{k22})^2 > (w^{''}_{k33})^2$, i.e., $(w^{''}_{k22})^2 = \text{Max}_{j=1,2,3}(w^{''}_{kjj})^2$: Then $\text{Max}_\theta B(\theta) = (w^{''}_{k22})^2$. Hence,

$$\text{Max}_{\theta,\phi}\sqrt{A(\theta,\phi)} = |w^{''}_{k22}|. \tag{36}$$

Subcase2:$(w^{''}_{k22})^2 < (w^{''}_{k33})^2$, i.e., $(w^{''}_{k11})^2 < (w^{''}_{k22})^2 < (w^{''}_{k33})^2$: Then $\text{Max}_\theta B(\theta) = (w^{''}_{k33})^2$. Hence,

$$\text{Max}_{\theta,\phi}\sqrt{A(\theta,\phi)} = |w^{''}_{k33}|. \tag{37}$$

So, combining all cases, we get:

$$\text{Max}_{\theta,\phi}\sqrt{A(\theta,\phi)} = \text{Max}^3_{j=1}|w^{''}_{kjj}|,\ k = 1,2. \tag{38}$$

So, the unsteerability criterion [Eq. (41)] turns out to be:

$$\text{Max}_{j=1,2,3}|w^{''}_{kjj}| \leq \frac{1}{2}. \tag{39}$$

Table 4 State parameters of each of the four conditional states are specified here

State	$\vec{X}_1$	$\vec{X}_2$	T
ρ^{00}_{AC}	Θ	Θ	$\text{diag}(w^{''}_{111}w^{''}_{211}, -w^{''}_{122}w^{''}_{222}, w^{''}_{133}w^{''}_{233})$
ρ^{01}_{AC}	Θ	Θ	$\text{diag}(-w^{''}_{111}w^{''}_{211}, w^{''}_{122}w^{''}_{222}, w^{''}_{133}w^{''}_{233})$
ρ^{10}_{AC}	Θ	Θ	$\text{diag}(w^{''}_{111}w^{''}_{211}, w^{''}_{122}w^{''}_{222}, -w^{''}_{133}w^{''}_{233})$
ρ^{11}_{AC}	Θ	Θ	$\text{diag}(-w^{''}_{111}w^{''}_{211}, -w^{''}_{122}w^{''}_{222}, -w^{''}_{133}w^{''}_{233})$

$\vec{X}_1, \vec{X}_2$ denote the local Bloch vectors corresponding to first and second party, respectively, whereas T denote the correlation tensor. diag$(*,*,*)$ stands for a diagonal matrix. Clearly each of the conditional state is in its canonical form [Eq. (22)]

where $k = 1,2$ correspond to states $\rho^{(2)}_{AB}$ and $\rho^{(2)}_{BC}$, respectively. So $\rho^{(2)}_{AB}$ and $\rho^{(2)}_{BC}$ and therefore ρ_{AB} and ρ_{BC} are unsteerable. Steerability of state remaining invariant under application of linear map [Eq. (20)], considering the canonical forms $\rho^{(2)}_{AB}$ and $\rho^{(2)}_{BC}$ as the initial states used in the network. Depending on the output of BSM obtained by Bob (and result communicated to Alice and Charlie), the conditional states shared between Alice and Charlie are given by ρ^{ij}_{AC}, $i,j = 0,1$ (see Table 4). $\forall i,j$, ρ^{ij}_{AC} has null local Blochs and diagonal correlation tensor.

Hence, for each of the conditional states, L.H.S. of Eq. (23) turns out to be:

$$\text{Max}_{x_1,x_2,x_3}\sqrt{\sum_{j=1}^{3}(x_j w^{''}_{1jj}w^{''}_{2jj})^2} \tag{40}$$

Following the same procedure of maximization as above, the optimal expression of the maximization problem [Eq. (40)] is given by:

$$\text{Max}_{j=1,2,3}|w^{''}_{1jj}w^{''}_{2jj}|$$

Using Eq. (39), the maximum value of Eq. (40) turns out to be $\frac{1}{4}$. Each of the four conditional states thus satisfies the unsteerability criterion [Eq. (23)]. So if both the initial states satisfy Eq. (23) and have null local Bloch vector (corresponding to first party), then none of the conditional states generated in the network is steerable. Hence genuine activation of steering does not occur. States satisfies Eq. (5).

Appendix C

Details of the numerical observation given in Sect. 6: Without loss of any generality, of two initial states, let ρ_{BC} has non-null Bloch vector corresponding to the first party, i.e., $\vec{u_1}=\Theta, \vec{u_2}\neq\Theta$. ρ_{BC} thus has the form:

$$\rho_{BC}=\frac{1}{4}(\mathbb{I}_{2\times 2}+\vec{\mathfrak{u}_2}.\vec{\sigma}\times\mathbb{I}_2+\mathbb{I}_2\otimes\vec{\mathfrak{v}_2}.\vec{\sigma} + \sum_{j_1,j_2=1}^{3} w_{2j_1j_2}\sigma_{j_1}\otimes\sigma_{j_2}),$$

After applying Λ [Eq. (20)] followed by local unitary operations, the canonical form $\rho_{AB}^{(2)}$ of ρ_{AB} is given by Eq. (27), whereas that of ρ_{BC} is given by:

$$\rho_{BC}^{(2)}=\frac{1}{4}(\mathbb{I}_{2\times 2}+\vec{\mathfrak{u}_2^{''}}.\vec{\sigma}\times\mathbb{I}_2+\sum_{j=1}^{3}w_{2jj}^{''}\sigma_j\otimes\sigma_j), \quad (41)$$

Now $\rho_{AB}^{(2)}$ and $\rho_{BC}^{(2)}$ both satisfy unsteerability criterion given by Eq. (23). This in turn gives:

$$\text{Max}_{x_1,x_2,x_3}\sqrt{\sum_{j=1}^{3}(x_j w_{1jj}^{''})^2}\leq\frac{1}{2} \quad (42)$$

and

$$\text{Max}_{x_1,x_2,x_3}(\vec{\mathfrak{u}_2^{''}}.\hat{x})^2+2\sqrt{\sum_{j=1}^{3}(x_j w_{2jj}^{''})^2}\leq 1 \quad (43)$$

with $\hat{x}=(x_1,x_2,x_3)$ denoting unit vector. While the closed form of Eq. (42) is given by Eq. (39) for $k=1$, the same for Eq. (43) is hard to derive owing to the complicated form of the maximization problem involved in it. Now as $\rho_{BC}^{(2)}$ satisfies an unsteerability criterion [Eq. (43)] so it is unsteerable and consequently violates Eq. (5):

$$\sum_{j=1}^{3}(w_{2jj}^{''})^2\leq 1 \quad (44)$$

As discussed above, the canonical forms $\rho_{AB}^{(2)}$ and $\rho_{BC}^{(2)}$ as the initial states used in the network. Depending on Bob's output, the conditional states shared between Alice and Charlie are given by ρ_{AC}^{ij}, $i,j=0,1$ (see Table 5).

Let us consider ρ_{AC}^{00}. Using state parameters (Table 5) of ρ_{AC}^{00}, L.H.S. of Eq. (23) becomes:

$$\text{Max}_{x_1,x_2,x_3}((x_1\mathfrak{u}_{21}^{''}w_{111}^{''}-x_2\mathfrak{u}_{22}^{''}w_{122}^{''}+x_3\mathfrak{u}_{23}^{''}w_{133}^{''})^2 + sqrt\sum_{j=1}^{3}(x_j w_{1jj}^{''}w_{2jj}^{''})^2), \quad (45)$$

Table 5 State parameters of each of the four conditional states are specified here

State	$\vec{X}_1$	$\vec{X}_2$	T
ρ_{AC}^{00}	$(w_{111}^{''}u_{21}^{''}, -w_{122}^{''}u_{22}^{''}, w_{133}^{''}u_{23}^{''})$	Θ	$\text{diag}(w_{111}^{''}w_{211}^{''}, -w_{122}^{''}w_{222}^{''}, w_{133}^{''}w_{233}^{''})$
ρ_{AC}^{01}	$(-w_{111}^{''}u_{21}^{''}, w_{122}^{''}u_{22}^{''}, w_{133}^{''}u_{23}^{''})$	Θ	$\text{diag}(-w_{111}^{''}w_{211}^{''}, w_{122}^{''}w_{222}^{''}, w_{133}^{''}w_{233}^{''})$
ρ_{AC}^{10}	$(w_{111}^{''}u_{21}^{''}, w_{122}^{''}u_{22}^{''}, -w_{133}^{''}u_{23}^{''})$	Θ	$\text{diag}(w_{111}^{''}w_{211}^{''}, w_{122}^{''}w_{222}^{''}, -w_{133}^{''}w_{233}^{''})$
ρ_{AC}^{11}	$(-w_{111}^{''}u_{21}^{''}, -w_{122}^{''}u_{22}^{''}, -w_{133}^{''}u_{23}^{''})$	Θ	$\text{diag}(-w_{111}^{''}w_{211}^{''}, -w_{122}^{''}w_{222}^{''}, -w_{133}^{''}w_{233}^{''})$

$\vec{X}_1, \vec{X}_2$ denote the local Bloch vectors corresponding to first and second party, respectively, whereas T denote the correlation tensor. diag$(*,*,*)$ stands for a diagonal matrix. Clearly each of the conditional state is in its canonical form [Eq. (22)]

where $\mathfrak{u}_{21}^{''},\mathfrak{u}_{22}^{''},\mathfrak{u}_{23}^{''}$ are the components of real valued vector Bloch vector $\vec{\mathfrak{u}_2^{''}}$. In Eq. (45), maximization is to be performed over x_1,x_2,x_3, whereas the state parameters are arbitrary. Now the expression in Eq. (45) is numerically maximized over all the state parameters involved and also x_1,x_2,x_3 under the following restrictions:

- $w_{111}^{''}\leq\frac{1}{2}$
- $w_{122}^{''}\leq\frac{1}{2}$
- $w_{133}^{''}\leq\frac{1}{2}$
- $\sum_{j=1}^{3}(w_{2jj}^{''})^2\leq 1$.

While the first three restrictions are due to the unsteerability of $\rho_{AB}^{(2)}$, i.e., given by Eq. (39) for $k=1$, the last restriction is provided by Eq. (44)(a consequence of unsteerability of $\rho_{BC}^{(2)}$). Maximum value of the above maximization problem [Eq. (45)] turns out to be 0.75, corresponding maxima (alternate maxima exists) given by $w_{111}^{''}=0.5$, $w_{122}^{''}=0.454199$, $w_{133}^{''}=0.46353$, $w_{211}^{''}=-1$, $w_{222}^{''}=0$, $w_{233}^{''}=0$, $\mathfrak{u}_{21}^{''}=1$, $\mathfrak{u}_{22}^{''}=0$, $\mathfrak{u}_{23}^{''}=0$, $x_1=1$, $x_2=0$ and $x_3=0$. Maximum value less than 1 implies that the original maximization problem [Eq. (45)], where maximization is to be performed only over x_1,x_2,x_3(for arbitrary state parameters) under the above restrictions (resulting from unsteerability of $\rho_{AB}^{(2)},\rho_{BC}^{(2)}$), cannot render optimal value greater than 1. Consequently conditional state ρ_{AC}^{00} satisfies the unsteerability criterion [Eq. (23)] and is therefore unsteerable. So, in case Bob's particles get projected along $|\phi^+\rangle$, genuine activation of steering does not occur in the linear network. In similar way, considering, other three conditional states, it is checked that the unsteerability criterion [Eq. (23)] is satisfied in each case. Genuine activation of steering is thus impossible for all possible outputs of Bob. Hence when one of the initial states has null local Bloch vector corresponding to first party, genuine activation of steering does not occur.

Appendix D

See Table 6.

Table 6 Bloch matrix parameters of each of the four conditional states generated in the linear swapping network using Ω_3, Ω_4 [Eq. (24)] as initial states are specified here

State	$\vec{U}$	$\vec{V}$	$\mathbf{T}$
ρ_{AC}^{00}	$(0, 0, 0.98107)$	Θ	diag$(0.0729052, -0.0729052, 0.0128697)$
ρ_{AC}^{01}	$(0, 0, 0.98107)$	Θ	diag$(-0.0729052, 00.0729052, 0.0128697)$
ρ_{AC}^{10}	$(0, 0, 0.907448)$	Θ	diag$(0.0729052, 0.0729052, 0 - 0.0128697)$
ρ_{AC}^{11}	$(0, 0, 0.907448)$	Θ	diag$(-0.0729052, -0.0729052, 0 - 0.0128697)$

$\vec{U}, \vec{V}$ denote the local Bloch vectors corresponding to first and second party, respectively, whereas $\mathbf{T}$ denote the correlation tensor. diag$(*, *, *)$ stands for a diagonal matrix. Now, for each conditional state, $\mathbf{T}$ being in diagonal form and $\vec{V} = \Theta$, for all i, j, ρ_{AC}^{ij} is in its canonical form [Eq. (22)]

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Correction

Correction: Revealing hidden steering nonlocality in a quantum network

Kaushiki Mukherjee[1,a], Biswajit Paul[2,b], and Soma Mandal[3,c]

[1] Department of Mathematics, Government Girls' General Degree College, Ekbalpore, Kolkata 700023, India
[2] Department of Mathematics, Balagarh Bijoykrishna Mahavidyalaya, Balagarh, Hooghly 712501, India
[3] Department of Physics, Government Girls' General Degree College, Ekbalpore, Kolkata 700023, India

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In this article the affiliation details for the author Soma Mandal were incorrectly given as 'Department of Mathematics, Government Girls' General Degree College, Ekbalpore, Kolkata 700023, India' but should have been 'Department of Physics, Government Girls' General Degree College, Ekbalpore, Kolkata 700023, India'.

The original article has been corrected.

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[a] e-mail: kaushiki.wbes@gmail.com (corresponding author)
[b] e-mail: biswajitpaul4@gmail.com
[c] e-mail: soma2778@gmail.com

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