Quantum Information Processing (2019) 18:212
https://doi.org/10.1007/s11128-019-2328-0

Restricted distribution of quantum correlations in bilocal network

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Received: 9 November 2018 / Accepted: 12 May 2019 / Published online: 18 May 2019

Abstract
Analyzing shareability of correlations arising in any physical theory may be considered as a fruitful technique of studying the theory. Our present topic of discussion involves an analogous approach of studying quantum theory. For our purpose, we have deviated from the usual procedure of assessing monogamous nature of quantum correlations in the standard Bell-CHSH scenario. We have considered correlations arising in a quantum network involving independent sources. Precisely speaking, we have analyzed monogamy of nonbilocal correlations by deriving a relation restricting marginals. Interestingly, restrictions constraining distribution of nonbilocal correlations remain same irrespective of whether inputs of the nodal observers are kept fixed (in different bilocal networks) while studying nonbilocal nature of marginal correlations.

Keywords Quantum correlations · Monogamy · Bell locality · Quantum network · Bilocality

1 Introduction

Entanglement and nonlocality, the two most intrinsic features of quantum theory, play ubiquitous role in analyzing departure of the theory from the classical world. While the former is a property of quantum states [1], the latter mainly characterizes nature of

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correlations arising due to measurements on quantum systems [2–4]. Considered to be two inequivalent resources in general, both of these features form the basis of various information processing tasks such as device-independent entanglement witnesses [5], quantum key distribution (QKD) [6–9], Bayesian game theoretic applications [10], private randomness generation [11,12], etc, which cannot be performed by any classical resource. One of the inherent features responsible for strengthening efficiency of quantum resources over classical ones is the existence of restrictions over shareability of quantum particles or quantum correlations in multiparty scenario [13–24].

Research activities conducted so far clearly point out the existence of limitations over shareability of both quantum nonlocality [13,15] and entanglement [14,20–24]. Such sort of limitations is frequently referred to as monogamy of nonlocality and entanglement, respectively. Precisely speaking, let a tripartite state be shared between three parties, say, Alice, Bob and Charlie. If Alice's qubit is maximally entangled with that of Bob, then neither the state shared between Alice and Charlie nor that between Bob and Charlie is entangled. Now consider the tripartite correlations generated due to measurements on a quantum system shared between Alice, Bob and Charlie. If the marginal correlations shared between any two parties, say Alice and Bob violate Bell-CHSH inequality [25] maximally then neither marginal shared between Alice and Charlie nor that shared between Bob and Charlie can show Bell-CHSH violation. However, no such restriction exists over shareability of classical correlations. Over years, different trade-off relations have been designed to capture monogamous nature of not only quantum correlations but also of correlations abiding by no signaling principle [26]. Our present topic of discussion is contributory in this direction. To be precise, we have explored shareability of correlations characterizing quantum *bilocal network*.

Over past few years, there has been a trend of studying quantum networks involving independent sources [27,28]. Network involving two independent sources is referred to as 'bilocal' network (see Fig. 1). It was first introduced in [27]. Since then study of quantum networks characterized with source independence has been subject matter of thorough investigations [29–38] due to multi-faceted utility of the source independence assumption both from theoretical and experimental perspectives such as lowering down restrictions for detecting quantumness (nonclassical feature) in a network via some notions of quantum nonlocality (different from the standard Bell nonlocality) [30,33]. Besides, it is found to be important to study detection loophole in some local models [39,40]. From experimental perspectives, source independent networks form basis of various experiments related to quantum information and communication such as various device-independent quantum information processing tasks [5,8–10], some communication networks dealing with entanglement percolation [41], *quantum repeaters* [42] and *quantum memories* [43], etc. Owing to the significance of these networks, study of correlations generated in such networks has gained immense importance. In this context, one obvious direction of investigation evolves around manifesting shareability of correlations in such networks. Our discussions will channelize in that direction.

To the best of authors' knowledge, research activities on monogamy of quantum correlations, conducted so far, basically consider the standard Bell scenario [13,15]. Here, we have shifted from that usual notion of Bell-CHSH nonlocality thereby exploring shareability of quantum correlations in spirit of nonbilocality [27,28]. Precisely speaking, we have considered quantum network involving two independent sources with an urge to investigate whether nonclassical feature of quantum correlations generated in such networks exhibit monogamy or not. We have obtained affirmative answer to this query which in turn points out the indifference between the two notions of nonlocality: Bell-CHSH nonlocality and nonbilocality in context of characterizing shareability of quantum correlations.

One may note that for studying monogamy in the standard Bell-CHSH scenario, it is assumed that the nodal party (for instance Alice in the example discussed before) has fixed measurement settings. For instance, to analyze Bell violation by each of two sets of bipartite correlations: $P(a, b|x, y)$, shared between Alice, Bob and $P(a, c|x, z)$, shared between Alice, Charlie (a, b, c and x, y, z denoting binary outputs and inputs of Alice, Bob and Charlie, respectively), Alice's measurement settings are assumed to be fixed. However, recently, in [44], a trade-off relation has been suggested giving restriction over upper bound of Bell-CHSH violation by all the possible bipartite marginals where the measurement settings of nodal party were not assumed to be fixed. Here, we have firstly derived a monogamy relation for nonbilocal correlations. Then, we have relaxed the assumption of fixed setting by nodal party, thereby designing a trade-off relation restricting the nonbilocal nature of the marginal correlations. Interestingly, nature of restrictions to exhibit nonbilocality by the marginals remains invariant irrespective of the assumption of fixed measurement settings of nodal party. Throughout the manuscript we have considered that each party has a two-dimensional quantum system under its control.

Rest of the article is organized as follows. First, we discuss some ideas motivating our work in Sect. 2. Next in Sect. 3, we give a brief review of the bilocal network scenarios and some results related to these scenarios which in turn will facilitate our further discussions. In Sect. 4, we first sketch the network scenario(s) in details. Depending on inputs and also outputs of some of the parties (involved in the scenario), we basically consider two scenarios. Then, we derive the monogamy relation in Sect. 4 followed by a trade-off relation in Sect. 5 restricting the correlations generated therein. Some practical implications of our findings have been discussed in Sect. 6. Finally, we have concluded in Sect. 7 discussing possible future directions of research activities.

2 Motivation

As already pointed out before, in recent times, study of quantum networks (with independent sources) has gained paramount interest [29–36]. So, in context of analyzing nonclassicality of quantum correlations in such networks, assessment of monogamous nature (if any) of the correlations is crucial for developing a better insight in related fundamental issues. Interestingly, from practical view point, existence of restrictions over shareability of quantum correlations is utilized to design quantum secret sharing protocol secure against eavesdropping [45–47]. To be specific, it is this nonclassical feature of quantum correlations that plays a vital role to provide security against external attack better than any classical protocol. So, if monogamous nature of correlations arising in quantum networks involving independent sources can be guaranteed then that will be definitely helpful for security analysis in secret sharing protocols involving such networks. So possible issues related with restricted shareability of quantum correlations in network scenario deserve detailed investigations. This basically motivates our current topic.

3 Preliminaries

3.1 Bilocal scenario

Bilocal network as designed in [27,28] is shown in Fig. 1. Out of three scenarios described in [28], here we consider two scenarios, namely P^{14} and P^{22} scenarios involving bilocal network. The network involves three parties Alice(A), Bob(B) and Charlie(C) and two sources $\mathbf{S}_1$ and $\mathbf{S}_2$. All the parties and sources are arranged in a linear fashion. A source is shared between any pair of adjacent parties. Sources $\mathbf{S}_1$ and $\mathbf{S}_2$ are independent to each other (*bilocal assumption*). A physical system represented by variables λ_1 and λ_2 is send by $\mathbf{S}_1$ and $\mathbf{S}_2$, respectively. Bob receives two particles (one from each of $\mathbf{S}_1$ and $\mathbf{S}_2$). Independence of λ_1 and λ_2 is ensured by that of $\mathbf{S}_1$ and $\mathbf{S}_2$. In both P^{14} and P^{22} scenarios, each of Alice and Charlie can perform dichotomic measurements on their systems. The binary inputs are denoted by $x, z \in \{0, 1\}$ for Alice and Charlie, and their outputs are labeled as $a, c \in \{0, 1\}$, respectively. Bob performs measurement on the joint state of the two systems that he receives from $\mathbf{S}_1$ and $\mathbf{S}_2$. In P^{22} scenario, Bob performs two dichotomic measurements $y \in \{0, 1\}$ having outputs $b \in \{0, 1\}$.

In P^{14} scenario, Bob performs single measurement y having 4 outputs $b = \overrightarrow{\mathbf{b}} = b_0b_1 = 00, 01, 10, 11$.

In both scenarios, the correlations obtained in the network are local if they take the form: $P(a, b, c|x, y, z) = \iint d\lambda_1 d\lambda_2 \rho(\lambda_1, \lambda_2) U$

$$\text{where } U = P(a|x, \lambda_1) P(b|y, \lambda_1, \lambda_2) P(c|z, \lambda_2) \tag{1}$$

where λ_1 characterizes the state of the bipartite system produced by the source S_1 and λ_2 for the system S_2. Tripartite correlations $P(a, b, c|x, y, z)$ are bilocal if they can be decomposed in above form (Eq. (1)) together with the restriction:

$$\rho(\lambda_1, \lambda_2) = \rho_1(\lambda_1)\rho_2(\lambda_2) \tag{2}$$

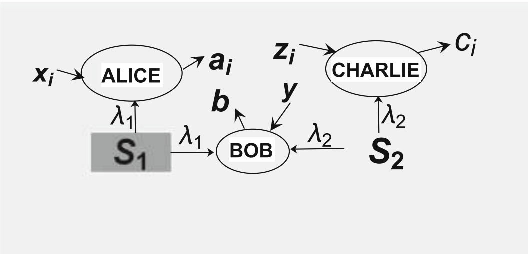

Fig. 1 Schematic diagram of a bilocal network [27,28]. In P^{22} scenario, y corresponds to two inputs of Bob y_0 and y_1 and b corresponds to two outputs b_0 and b_1. In P^{14} scenario, y corresponds to single input of Bob and b corresponds to 4 outputs $b_0b_1 = 00, 01, 10, 11$

imposed on the probability distributions of the hidden variables λ_1, λ_2. Eq. (2) refers to the *bilocal constraint*. Tripartite correlations of the form (Eq. (1)) and (Eq. (2)) are bilocal if they satisfy the inequality [28]:

$$\sqrt{|I|} + \sqrt{|J|} \leq 1 \tag{3}$$

Terms appearing in above equation are discussed in Table 1. Denoting $\sqrt{|I|} + \sqrt{|J|}$ as $\mathbf{B}$, Eq. (3) becomes:

$$\mathbf{B} \leq 1 \tag{4}$$

Clearly, violation of Eq. (4) acts as a sufficient criterion for detecting nonbilocality of corresponding correlations. In [36,37], referring $\mathbf{B}$ as *bilocality parameter*, an upper bound of quantum violation of the bilocal inequality (Eq. (4)) has been derived for both P^{22} scenario [36] and P^{14} scenario [37]. We next briefly review scenarios considered in [36,37] along with some of the related findings which will be used later in course of our work.

3.2 Bilocal quantum network [27,28]

Let each of $\mathbf{S}_1$ and $\mathbf{S}_2$ sends a two qubit quantum state. Let $\mathbf{S}_1$ sends ρ_{AB} to Alice and Bob, whereas $\mathbf{S}_2$ sends ρ_{BC} to Bob and Charlie. In general, any bipartite state density matrix representing a quantum state (ρ, say) can be defined as:

$$\rho = \frac{1}{2^2} \sum_{i_1, i_2 = 0}^{3} T_{i_1 i_2} \sigma_{i_1}^1 \bigotimes \sigma_{i_2}^2 \tag{5}$$

with σ_0^k, denoting the identity operator in the Hilbert space of k^{th} qubit and $\sigma_{i_k}^k$, denote the Pauli operators along three mutually perpendicular directions, $i_k = 1, 2, 3$. The entries of the correlation matrix T_ρ of ρ denoted by $T_{i_1 i_2}$ are real and given by:

$$T_{i_1 i_2} = \mathrm{Tr}[\rho \sigma_{i_1}^1 \bigotimes \sigma_{i_2}^2],\ i_1, i_2 \in \{1, 2, 3\}. \tag{6}$$

Let T_{AB} and T_{BC} denote correlation tensor of state ρ_{AB} and ρ_{BC}, respectively, and let upper bound of violation of the bilocal inequality (Eq. (4)) be given by $\mathbf{B}_{\mathrm{Max}}$. Recently, $\mathbf{B}_{\mathrm{Max}}$ has been derived for both the scenarios [36,37]. Interestingly, irrespective of variation in measurement settings of Bob (see Table 2), closed form of $\mathbf{B}_{\mathrm{Max}}$ turns out to be same:

$$\mathbf{B}_{\mathrm{Max}} = \sqrt{\sum_{i=1}^{2} \sqrt{\omega_i^A * \omega_i^C}},\ \ \omega_1^{A(C)} > \omega_2^{A(C)}, \tag{7}$$

with ω_1^A and ω_2^A (ω_1^C and ω_2^C) are the larger two eigenvalues of $T_{\mathrm{AB}}^T T_{\mathrm{AB}}$ ($T_{BC}^T T_{BC}$).

Table 1 Details of the terms appearing in Eq. (3) in both the scenarios [27,28]. A_x and C_z stand for the observables corresponding to binary inputs x and z of Alice and Charlie, respectively. $a, c \in \{0, 1\}$ denote the corresponding outputs

Scenario	I and J	Expressions of correlators	Observables and outputs of Bob
P^{22}	$I = \frac{1}{4} \sum_{x,z=0,1} \langle A_x B_0 C_z \rangle$	$\langle A_x B_y C_z \rangle = \sum_{a,b,c} (-1)^{a+b+c} P(a, b, c \vert x, y, z)$	B_y:observable corresponding to binary inputs y of Bob with outputs $b \in \{0, 1\}$.
	$J = \frac{1}{4} \sum_{x,z=0,1} (-1)^{x+z} \langle A_x B_1 C_z \rangle$		
P^{14}	$I = \frac{1}{4} \sum_{x,z=0,1} \langle A_x B^0 C_z \rangle$	$\langle A_x B^y C_z \rangle = \sum_{a,b_0b_1,c} (-1)^{a+b_y+c} P(a, b_0b_1, c \vert x, z)$	B^y:observable corresponding to a single input 4 outputs: $\overrightarrow{\mathbf{b}} = b_0b_1 =$ 00, 01, 10, 11.
	$J = \frac{1}{4} \sum_{x,z=0,1} (-1)^{x+z} \langle A_x B^1 C_z \rangle$		

Table 2 The table provides the measurement settings of Bob for both the scenarios. $\mathbf{fi_i}^A$ denotes the direction along which Bob performs projective measurement on the particle that it receives from source S_1 and $\mathbf{fi_i}^D$ denotes the direction along which Bob performs projective measurement on the particle that it receives from source S_2. Alice and Charlie perform projective measurements in two arbitrary directions: $_{\mathbf{i}}$.œ and $_{\mathbf{i}}$.œ$(i = 0, 1)$, respectively [36,37]

Scenario	Bob's measurement settings
P^{22} [36]	$\mathbf{fi_i}^A$.œ $\otimes$ $\mathbf{fi_i}^D$.œ$(i = 0, 1)$
	i.e., Bob performs separable measurements on joint state of two qubits (received from $\mathbf{S}_1$ and $\mathbf{S}_2$)
P^{14} [37]	Bob performs full Bell-state measurements, i.e., he measures joint state of two qubits in
	Bell-basis. Projection of the joint state in a Bell state thereby corresponding to an output (total 4)

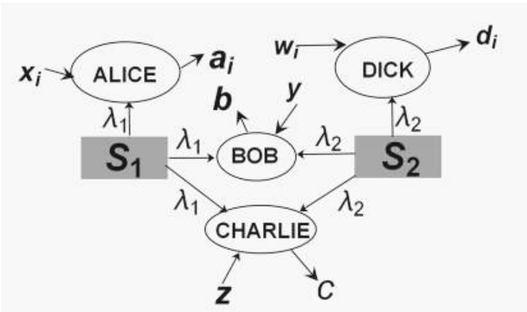

Fig. 2 Schematic diagram of network $\mathcal{N}$. In P^{22} scenario y, z corresponds to two inputs of Bob y_0 and y_1 and two inputs of Charlie z_0 and z_1, respectively. b corresponds to two outputs b_0 and b_1 of Bob, and likewise c denotes two outputs of Charlie c_0, and c_1. In P^{14} scenario y, z corresponds to Bell-state measurement (single input) of Bob and Charlie, respectively. b corresponds to 4 outputs: $b_0 b_1 = 00,\ 01,\ 10,\ 11$ according to projection of the joint state of two qubits in Bob's control in Bell state $|\phi^+\rangle$, $|\phi^-\rangle$, $|\psi^+\rangle$ and $|\psi^-\rangle$, respectively. Analogously 4 outputs of Charlie are denoted by $c_0 c_1 = 00,\ 01,\ 10,\ 11$

After discussing the mathematical pre-requisites, we proceed to present our findings.

4 Nonbilocal monogamy

For our purpose, we have considered P^{22} scenario and P^{14} scenario under the assumption that Bob performs Bell-state measurement (as stated in Table 2). For exploring restriction(if any) over shareability of nonbilocal correlations, first we define the network (Fig. 2) which will encompass both the scenarios.

4.1 Quantum network scenario

Consider a network ($\mathcal{N}$) involving four parties Alice, Bob, Charlie, Dick and two independent sources $\mathbf{S}_1$ and $\mathbf{S}_2$. Each of the two sources generates a tripartite quantum state. Let $\mathbf{S}_1$ generates ρ_{ABC}, sending a qubit to each of Alice, Bob and Charlie. Analogously, $\mathbf{S}_2$ generates ρ_{BCD}, sending a qubit to each of Bob, Charlie and Dick. So, each of Bob and Charlie receives two qubits, whereas remaining two parties receives one qubit each. Let Bob and Charlie be referred to as *intermediate parties*, whereas Alice and Dick be referred to as *extreme parties*. The extreme parties perform arbitrary projective measurements locally on their qubits. Inputs of Alice and Dick are labeled as $x, w \in \{0, 1\}$ and outputs as $a, d \in \{0, 1\}$, respectively. Each of two intermediate parties Bob and Charlie performs measurements on joint state of its two qubits. In P^{22} scenario, each of Bob and Charlie performs dichotomic measurements, i.e., $y, z \in \{0, 1\}$ with corresponding outputs $b, c \in \{0, 1\}$, whereas in P^{14} scenario each of them can perform a single measurement having four outputs (as discussed in Sect. 3). We consider some specific forms of measurements for Bob and Charlie in both the scenarios. While in P^{22} scenario, each of Bob and Charlie can perform separable measurements [36], in P^{14} scenario each of them performs Bell-state measurement (BSM) [37] on the joint state of the two qubits sent by the sources. In both these scenarios, four partite correlation terms $P(a, b, c, d|x, y, z, w)$, arising due to measurements by the parties on their respective qubits characterize the network ($\mathcal{N}$). Let W_B and W_C denote the set of tripartite marginals $P(a, b, d|x, y, w)$ and $P(a, c, d|x, z, w)$, respectively. Now W_B can be interpreted as the set of tripartite correlations arising due to binary measurements by each of three parties Alice, Bob and Dick in a network involving two independent sources $\mathbf{S}_1$ and $\mathbf{S}_2$. Hence, correlations from W_B characterize the bilocal network ($\mathcal{N}_B$, say) involving Alice, Bob and Dick. Analogously, W_C characterizes bilocal network ($\mathcal{N}_C$, say) involving parties Alice, Charlie and Dick. Each of the two bilocal networks $\mathcal{N}_B$ and $\mathcal{N}_C$ may be referred to as a *reduced network* obtained from the original bilocal network $\mathcal{N}$. Clearly, extreme parties of $\mathcal{N}$ are common in both the reduced networks ($\mathcal{N}_B$, $\mathcal{N}_C$) and may be referred to as the *nodal parties*. In Fig. 3, we give a flowchart to design the scenario involved herein.

Now, we put forward the monogamy relation restricting nonbilocality of the tripartite marginals $P(a, b, d|x, y, w)$ and $P(a, c, d|x, z, w)$ in both the scenarios.

4.2 Restrictions over quantum correlations

Theorem 1 *Under the assumption that each of the extreme parties performs separable measurements or Bell-state measurements, if $\mathbf{B}^B_{\text{Max}}$ and $\mathbf{B}^C_{\text{Max}}$ denote the upper bound of violations of bilocal inequality (Eq. (4)) by correlations $P(a, b, d|x, y, w)$ and $P(a, c, d|x, z, w)$, respectively, then,*

$$(\mathbf{B}^B_{\text{Max}})^2 + (\mathbf{B}^C_{\text{Max}})^2 \leq 2. \tag{8}$$

Proof This theorem basically follows from the proof of Theorem 2 which will be discussed shortly.

Compared to a single nodal (common) party in standard Bell scenario, here, measurement settings of both the nodal parties (Alice and Dick) are kept fixed in order to sketch the monogamy relation (Eq. (8)). Alice and Dick's fixed measurement settings mainly refer to the fact each of their measurement settings remain unchanged

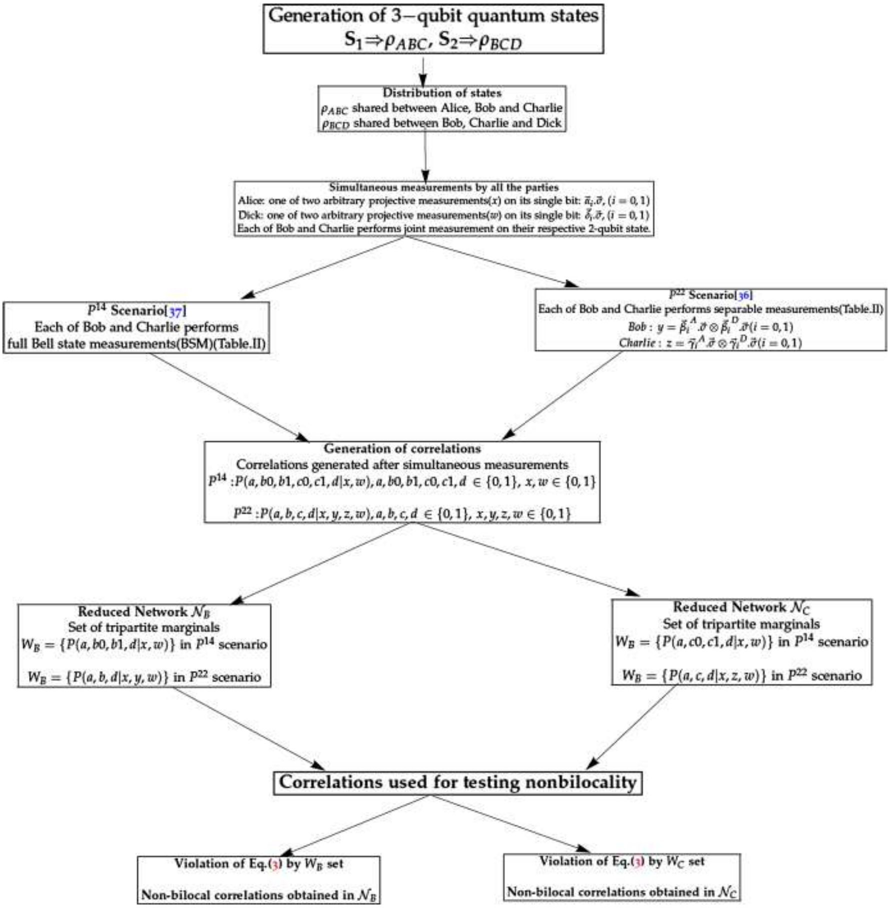

Fig. 3 A flowchart underlying our scheme of testing correlations is provided thereby listing all the steps involved therein. Due to nonbilocal monogamy (or trade-off relation) given by Eq. (8), any one of the two observations given in the two blocks at the end of the chart occurs. For deriving monogamy relation, Alice and Dick's measurement settings are kept fixed while collecting correlations in W_B and W_C. For sketching trade-off relation, no such restriction is imposed over their settings

in both the reduced networks $\mathcal{N}_B$, $\mathcal{N}_C$. To be precise, if Alice(Dick) performs measurement $M_A(M_D)$ in network $\mathcal{N}_B$, then in network $\mathcal{N}_C$ also measurement setting of Alice(Dick) is $M_A(M_D)$. As $\mathbf{B}_{\text{Max}}$ has the same form(Eq. (7)) in both P^{14} and P^{22} scenarios, so the inequality imposing restrictions on correlations in the two scenarios remains invariant.

Tightness of the constraint: By tightness of the monogamy relation given by Eq. (8), we interpret the existence of quantum correlations reaching the upper bound 2. For a particular instance, let each of the two sources $\mathbf{S}_1$ and $\mathbf{S}_2$ generates identical copy of a W state [48]:

$$|\Psi\rangle = \cos\mu_0|001\rangle + \sin\mu_1\sin\mu_0|010\rangle + \sin\mu_0\cos\mu_1|100\rangle \tag{9}$$

where $\mu_i(i=0,1) \in [0, \frac{\pi}{2}]$. Let, for $\mu_0 = \frac{\pi}{2}$, identical copies of the corresponding state($|\Psi\rangle\langle\Psi|$) are used in the network ($\mathcal{N}$). Let 1st, 2nd and 3rd qubit of one copy are sent to Alice, Bob and Charlie, respectively, whereas 1st, 2nd and 3rd qubit of second copy are sent to Dick, Bob and Charlie, respectively. Using the upper bound of violation given by Eq. (7) we get $(\mathbf{B}^B_{\text{Max}})^2 + (\mathbf{B}^C_{\text{Max}})^2 = \sqrt{(\text{Max}[0, \cos^2(2\mu_1)])^2} + \sqrt{(\text{Max}[1, \sin^2(2\mu_1)])^2} + \sqrt{(\text{Min}[0, \cos^2(2\mu_1)])^2} + \sqrt{(\text{Min}[1, \sin^2(2\mu_1)])^2}$. Clearly on simplification, $(\mathbf{B}^B_{\text{Max}})^2 + (\mathbf{B}^C_{\text{Max}})^2 = 2$.

Before discussing any further observation, we first put forward a lemma. □

Lemma *Under the assumption that Bob performs separable measurements in P^{22} scenario [36] and Bell-state measurement (BSM) in P^{14} scenario [37], maximal violation of the bilocal inequality (Eq. 4) is $\sqrt{2}$, maximum being taken over all possible quantum states .*

Proof From upper bound of violation of bilocal inequality (Eq. (4)) given by Eq. (7),

$$\mathbf{B}^2_{\text{Max}} = \sum_{i=1}^{2} \sqrt{\omega_i^A * \omega_i^C}. \tag{10}$$

By Cauchy–Schwarz's inequality, we get

$$\mathbf{B}^2_{\text{Max}} \leq \sqrt{\omega_1^A + \omega_2^A}\sqrt{\omega_1^C + \omega_2^C} \tag{11}$$

Now $\sqrt{\omega_1^A + \omega_2^A} = M(\rho_{\text{AB}})$ and likewise $\sqrt{\omega_1^C + \omega_2^C} = M(\rho_{\text{BC}})$ where $2M(\rho)$ denote maximal violation of Bell-CHSH inequality by a quantum state ρ [49]. Again, maximal possible quantum violation of Bell-CHSH is given by $2\sqrt{2}$, referred to as Tsirelson's bound [50]. Hence, maximal possible quantum violation of the bilocal inequality (Eq. (4)) turns out to be $\sqrt{2}$. □

The monogamy relation (Eq. (8)) puts restrictions over distribution of nonbilocal correlations among the two networks $\mathcal{N}_B$ and $\mathcal{N}_C$ in both P^{14} and P^{22} scenarios. To be precise, maximal violation of bilocal inequality (Eq. (4)), being $\sqrt{2}$, both of $(B^B_{\text{Max}})^2$ and $(B^C_{\text{Max}})^2$ could have been 2. But, this becomes impossible due to the restriction imposed by Eq. (8). Moreover, if any one set of tripartite marginals, say $P(a, b, d|x, y, w)$ (W_B set) shows violation of the bilocal inequality, the others set (W_C) of marginals does not violate the bilocal inequality. So, generation of nonbilocal correlations in one reduced network ($\mathcal{N}_B$, say) guarantees (considering generation of nonbilocal correlations up to detection by the sufficient criterion provided by violation of the bilocal inequality (Eq. (4))) the absence of any such nonclassical feature (nonbilocality) of quantum correlations in the other reduced network system ($\mathcal{N}_C$). Hence, if maximal violation of bilocal inequality is observed in one reduced network ($\mathcal{N}_B$, say), then correlations from $\mathcal{N}_C$ cannot violate the bilocal inequality (Eq. (4))) and hence may not be nonbilocal. Such an observation is analogous to existing results related to monogamy of quantum entanglement and nonlocality (standard Bell-CHSH sense).

5 Nonbilocal trade-off relation

Monogamy relation (Eq. (8)) guarantees existence of restrictions over distribution of nonbilocal correlations in reduced bilocal network systems. As already mentioned in the previous section, analogous to monogamy of nonlocal correlations in standard Bell-CHSH sense, measurement settings of nodal parties are kept fixed (in both the scenarios) for assessing monogamy of nonbilocal correlations. However, in [44], it was pointed out that comparison of monogamy and trade-off relations of nonlocal correlations guarantees relaxation of restrictions over shareability of nonlocal correlations among bipartite reduced states. Such an observation is quite intuitive owing to the fact that in contrast to fixed measurement settings of the nodal party (considering nature of the bipartite marginals for sketching monogamy relation), for giving trade-off relation (connecting amount of Bell-CHSH violation by the bipartite marginals), the measurement settings for the nodal parties are not considered to be invariant. Hence, optimization over parameters characterizing inputs of the nodal parties is possibly separate while considering Bell-CHSH violation by each of the reduced states. For instance, it may so happen that Bell-CHSH violation by reduced state ϱ_{AB} (obtained from state ϱ_{ABC}), is optimized for one measurement direction ($\mathbf{x_0}$, say) of Alice while the same by reduced state ϱ_{AC} is optimized for some other measurement direction ($\mathbf{x_1} \neq \mathbf{x_0}$, say) of Alice.

In this context, one may expect to encounter analogous observations in case of characterizing shareability of nonbilocal correlations. However, our findings guarantee somewhat counterintuitive feature. To be precise, relaxation of the restriction over nodal parties to have fixed measurement settings in both reduced networks fails to relax restrictions over shareability of nonbilocal correlations (up to existing sufficient criterion of detection of nonbilocality, i.e., violation of Eq. (3)). Also, proof of Theorem.1 turns out to be a special case of the proof of the following theorem.

Theorem 2 *In both P^{14} (Bob and Charlie performing Bell-state measurement) and P^{22} scenario (Bob and Charlie performing separable measurements), trade-off relation satisfied by upper bound of violation of bilocal inequality (Eq. (4)) by tripartite marginals in reduced networks $\mathcal{N}_B$ and $\mathcal{N}_C$ is same as the monogamy relation given by Eq. (8).*

Proof Each of Bob and Charlie performs separable measurements in P^{22} scenario, whereas perform Bell-state measurement in P^{14} scenario. In either of these scenarios, measurement settings of the nodal parties Alice and Dick may vary while considering violation of bilocal inequality in each of the two reduced networks ($\mathcal{N}_B$ and $\mathcal{N}_C$). Hence, by Eq. (11), we get:

$$(\mathbf{B}^B_{\mathrm{Max}})^2 + (\mathbf{B}^C_{\mathrm{Max}})^2 \leq \sqrt{\iota^B_1 + \iota^B_2}\sqrt{\Lambda^B_1 + \Lambda^B_2} + \sqrt{\iota^C_1 + \iota^C_2}\sqrt{\Lambda^C_1 + \Lambda^C_2} \tag{12}$$

where $\Lambda^B_1 \geq \Lambda^B_2 \geq \Lambda^B_3$, $\Lambda^C_1 \geq \Lambda^C_2 \geq \Lambda^C_3$, $\iota^B_1 \geq \iota^B_2 \geq \iota^B_3$ and $\iota^C_1 \geq \iota^C_2 \geq \iota^C_3$ are the eigen values of $T^T_{\mathrm{AB}}T_{\mathrm{AB}}$, $T^T_{\mathrm{AC}}T_{\mathrm{AC}}$, $T^T_{\mathrm{BD}}T_{\mathrm{BD}}$ and $T^T_{\mathrm{CD}}T_{\mathrm{CD}}$, respectively. Applying

A.M. $\geq$ G.M. over the positive terms $\iota_1^B + \iota_2^B$, $\Lambda_1^B + \Lambda_2^B$; $\iota_1^C + \iota_2^C$, and $\Lambda_1^C + \Lambda_2^C$, we get,

$$\frac{\iota_1^B + \iota_2^B + \Lambda_1^B + \Lambda_2^B + \iota_1^C + \iota_2^C + \Lambda_1^C + \Lambda_2^C}{2}.$$

Now for the state ρ_{ABC}(generated by the source $\mathbf{S}_1$) which is shared between Alice, Bob and Charlie [51],

$$\Lambda_1^B + \Lambda_2^B + \Lambda_1^C + \Lambda_2^C \leq 2, \tag{13}$$

Analogously, for the state ρ_{BCD}(generated by the source $\mathbf{S}_2$), we get,

$$\iota_1^B + \iota_2^B + \iota_1^C + \iota_2^C \leq 2, \tag{14}$$

Using, Eqs.(13,14), in Eq. (12), we get the required relation(Eq. (8)). □

Proof of Theorem 1 from that of Theorem 2 As has already been pointed out in the proof of Theorem 2, nodal parties are free to choose different measurement settings in the two reduced networks $\mathcal{N}_B$ and $\mathcal{N}_C$. Let M_A^B and M_A^C denote measurement settings of Alice in reduced network $\mathcal{N}_B$ and $\mathcal{N}_C$, respectively. Likewise, let M_D^B and M_D^C denote measurement settings of the other nodal party Dick in reduced network $\mathcal{N}_B$ and $\mathcal{N}_C$, respectively. So, quite obviously, $M_A^B = M_A^C$ and $M_D^B = M_D^C$ came as a special case of all possible measurement settings of Alice (M_A^B, M_A^C) and Dick (M_D^B and M_D^C). But this special case corresponds to the restriction of fixed measurement settings of nodal observers that is usually imposed for deriving monogamy relations and hence corresponds to the assumptions of Theorem 1. This in turn guarantees assumptions involving nodal observers' inputs in Theorem 1 as a subcase of the assumptions over nodal observers' inputs in Theorem 2. Consequently, proof of Theorem 1 is following from that of Theorem 2 and hence, monogamy relation and trade-off relation restricting shareability of nonbilocal correlations are the same (Eq. (8)). □

Tightness of trade-off relation: Assumptions over nodal observers' inputs in Theorem.1 being a sub case of that in Theorem 2, tightness of trade-off relation follows immediately from tightness of monogamy relation (Eq. (8)).

The trade-off relation, being of the same form as that of the monogamy relation, restriction over shareability of nonclassical feature of quantum correlations (in terms of nonbilocality) is the same irrespective of whether measurement settings of the nodal parties remain fixed or not. Recent study on Bell-CHSH nonlocality reveals analogous findings regarding the fact that restrictions over distribution of nonclassical quantum correlations are independent of the fact whether nodal party's inputs are fixed or not. To be specific in [51], a trade-off relation restriction shareability of nonlocal quantum correlations (Bell-CHSH) has been given which has the same form as that of a monogamy relation of Bell-CHSH nonlocality which was previously given in [44].

As has already been mentioned before, one should note that all our observations are applicable for qubits. This is a consequence of the fact that the network (see Fig. 2) involves sources which generate three qubit states. Whether our observations hold for higher dimensional quantum systems is an area of open research work.

After ensuring existence of restriction over shareability of nonclassical correlations in quantum network scenario (characterized by source independence), we now discuss below practical significance of monogamy of nonbilocal correlations.

6 Practical implication

Establishing existence of restrictions over shareability of correlations in bilocal networks involving quantum systems, we now proceed to justify that such monogamous nature of nonbilocal correlations may be used to provide security in quantum secret key sharing protocol (actual design of any such protocol is however yet to be accomplished). Before starting the analysis, we briefly review an existing result involving the use of monogamy of the standard Bell nonlocality in such a protocol.

In [46], Barrett et.al. proved a connection between the possibility of existence of a protocol secure against postquantum eavesdropping and quantum violation of Bell-CHSH inequality under nosignaling assumption. To be precise, they designed a protocol involving two parties (Alice and Bob, say). Alice and Bob share identical copies of entangled states. At the end of the protocol, a secret bit is generated in between Alice and Bob although the source, generating the entangled states, is controlled by eavesdropper. Security of such a protocol is based on monogamy of Bell-CHSH violation by quantum correlations. Though unable to design any such protocol involving bilocal network, we only give justification in support of our claim that nonbilocal monogamy can provide security analogously.

Consider a secret key sharing protocol based on a quantum network involving three parties Alice, Bob and Dick and two independent sources $\mathbf{S}_1$ and $\mathbf{S}_2$ each of which generates an entangled state. Alice, Bob and Dick are referred to as trusted parties. Let the parties perform simultaneous measurements on their respective quantum particles that they receive from the sources (analogous to Fig. 1). Entangled state being generated by each of the two sources, correlations obtained after simultaneous measurements of the parties (Bob performing separable measurements) are nonbilocal in nature [36]. Now, it may happen that the two sources $\mathbf{S}_1$ and $\mathbf{S}_2$ are under the control of an eavesdropper (untrusted party). Such a situation may be interpreted in the form of the sources generating tripartite quantum state where one bit from each source is under the control of the eavesdropper (schematic diagram involved herein is analogous to that of our scenario, Fig. 2, where Charlie may be considered as the eavesdropper). Now, the eavesdropper also performs measurement on his qubits. Depending on his control over the sources, the eavesdropper will get access to the key generated in the protocol. If he can get information about the secret key then the correlations shared between Alice, Dick and the eavesdropper will be nonbilocal (can be interpreted as correlations in reduced network $\mathcal{N}_C$ in our scenario). But in such a case, due to monogamous nature of nonbilocal correlations, the correlations shared between Alice, Bob and Dick (analogous to correlations in reduced network $\mathcal{N}_B$) will be bilocal (up to testing of bilocal inequality (Eq. (3))). Based on this observation, the trusted parties can detect the presence of the eavesdropper and thereby discard the protocol. On the contrary, if eavesdropper cannot gain any information about the key then the correlations shared between Alice, Dick and Charlie will no longer be nonbilocal and consequently nonbilocal correlations will be generated between the trusted parties. On getting nonbilocal correlations, they are assured that secrecy of the protocol is maintained. So monogamy of nonbilocality can be applied to design a secret bit sharing protocol involving quantum network secured against attacks of eavesdropper. Below, we give justification in support of our claim.

Now in our network scenario (Fig. 2), monogamy of nonbilocal correlations involves Bell violation by reduced states ρ_{AB}, ρ_{AC}, ρ_{BD} and ρ_{CD}. So, restrictions over distribution of nonbilocal correlations(due to sufficient criterion of nonbilocality given by violation of bilocal inequality given by Eq. (3)) in reduced networks $\mathcal{N}_B$ and $\mathcal{N}_C$ involve restrictions over shareability of nonlocal correlations (in sense of Bell-CHSH violation) among the reduced states. Framing of the secret key sharing protocol, being based on our network scenario, security of the protocol thus ultimately rests upon monogamous nature of the standard Bell-CHSH nonlocal correlations. Now, as already discussed before, there exists secret quantum key sharing protocol where security is guaranteed by monogamy of Bell-CHSH nonlocality [46]. This gives an indication about the possibility of designing a protocol, secured against eavesdropper's attack, via which secret bit can be generated in bilocal network scenario involving Alice, Bob (performing separable measurements or Bell-state measurement) and Dick even if any eavesdropper (Charlie) who is capable of performing any separable measurement or Bell-state measurement have control over both the independent sources $\mathbf{S}_1$ and $\mathbf{S}_2$.

7 Discussions

Over years, there has been thorough investigation of monogamous nature of quantum entanglement and quantum nonlocality in the standard Bell-CHSH scenario. In this paper, we have considered bilocal quantum network to investigate the same for some weaker form of quantum nonlocality (nonbilocality). Exploitation of our observations ensures monogamous nature of nonbilocal quantum correlations (up to existing detection criterion for nonbilocality (Eq. (4)))). For our purpose, we have considered P^{22} and P^{14} scenarios. Interestingly, restrictions over shareability of distribution of nonbilocal correlations among reduced networks ($\mathcal{N}_B$ and $\mathcal{N}_C$) are the same irrespective of the inputs of the nodal parties (Alice and Dick) remaining fixed or not for observing violation of bilocal inequality in the reduced networks individually. From our discussions so far, it can be safely concluded that under the assumption of Bob and Charlie performing separable measurements or Bell-state measurement (BSM), if quantum correlations in one reduced network ($\mathcal{N}_B$, say) exhibit nonbilocality, then the correlations from the other one ($\mathcal{N}_C$) cannot violate the bilocal inequality (Eq. (4)) and hence may be bilocal. As we have already discussed, such monogamous nature of nonbilocal correlations can be utilized to design a secret sharing protocol secure against eavesdropper's attack. However, we have not been able to explicitly design any such protocol. One may find interest to develop one such protocol involving bilocal network. Also, the assumption that an eavesdropper can perform only separable measurements or Bell-state measurement is a weaker assumption as there may be cases where an eavesdropper can perform more general measurements. For those cases, one may try to design monogamy relation of nonbilocal correlations under the assumption that Bob and Charlie can perform more general measurements. One may try to explore the utility of bilocal monogamy in quantum dialog protocols [52,53]. However, in [52,53], continuous variable systems are involved in quantum dialog protocol but till date the bilocal scenarios involve only discrete quantum systems. So, in order to utilize restrictions over shareability of nonbilocal correlations in such protocols, criterion to detect nonbilocal correlations must be established following which one may explore application (if any) of monogamous nature of nonbilocality in all such protocols. Also, study investigating inter-relation between monogamy of nonbilocality and other nonclassical aspects of quantum theory is a potential source of research activities.

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