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Characterizing quantum correlations in a fixed-input n-local network scenario

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Contrary to the Bell scenario, quantum nonlocality can be exploited even when all the parties do not have freedom to select inputs randomly. Such manifestation of nonlocality is possible in networks involving independent sources. One can utilize such a feature of quantum networks for purpose of entanglement detection of bipartite quantum states. In this context, we characterize correlations simulated in networks involving a finite number of sources generating quantum states when some parties perform fixed measurement. Beyond bipartite entanglement, we inquire the same for networks involving sources now generating pure tripartite quantum states. Interestingly, here also randomness in input selection is not necessary for every party to generate nonlocal correlation.

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I. INTRODUCTION

Entanglement of multipartite quantum systems [1] plays a pragmatic role in manifesting deviation of quantum theory (QT) from the classical world. Bell used this intrinsic feature of the theory to abandon the possibility of existence of any local realistic interpretation of QT [2,3] which, however, respects the no-signaling principle. Bell's theorem provides an empirical methodology to detect nonlocal behavior of quantum correlations (often referred to as Bell nonlocality), an experimental demonstration of which has already been provided [4,5].

Speaking of tests demonstrating Bell nonlocality, the most simple test was proposed by Clauser *et al.* [6]. Such a test involves two distant observers (Alice and Bob, say) such that each of them performs one binary measurement choosing randomly from a set of two measurements. To be precise, Alice randomly chooses one input from a set of two inputs ($\{\mathcal{A}_0, \mathcal{A}_1\}$, say) and similarly Bob randomly chooses input from another set, say $\{\mathcal{B}_0, \mathcal{B}_1\}$. Moreover, the choice of inputs of Alice does not depend on that of Bob and vice versa (measurement independence). Bipartite correlations generated after measurements are used in testing correlator-based inequality, more commonly referred to as CHSH inequality. Violation of CHSH inequality indicates nonlocal nature of corresponding correlations. To date, analogous to CHSH inequality [6], different correlator-based inequalities (referred to as Bell inequalities) have been derived. Detecting quantum nonlocality by any of these tests requires randomness in the choice of inputs of both the observers present in the corresponding measurement scenario. However, random selection of inputs by all observers is not a necessity to exploit nonclassicality of quantum correlations simulated in network scenarios [7–12] characterized by source independence (often referred to as n-local networks).

n-local quantum networks [13–19] basically refer to a network of n sources, independent of each other, such that each of these sources generates an m-partite quantum state ($m \geqslant 2$) shared between m distinct parties. Nonlocality of correlations generated in such networks was first observed in a bilocal ($n, m = 2$) network [13,14] where entanglement was distributed from two independent sources. Such type of nonlocality is referred to as nonbilocality [13,14]. Nonbilocality, or more general non-n-locality, differs from the usual sense of Bell nonlocality (standard nonlocality) where entanglement is distributed from a common source. Some of the measurement scenarios involved in n-local networks have been proposed where some (P_{14} or P_{13} measurement scenarios [13]) or all [8,9] of the observers perform a single measurement (referred to as "fixed measurement" [9]). All of these studies basically analyzed some specific instances of quantum non-n-locality in such measurement scenarios where not all observers [13] can randomly select inputs, thereby manifesting instances of "quantum nonlocality without inputs" [9]. Now, observation of quantum nonlocality in networks can be used for the purpose of detection of quantum entanglement in the same. In this context, we first intend to exploit quantum nonlocality in networks, characterized by source independence and fixed input criterion (for at least one of the observers). Quantum networks witnessing non-n-locality can then be used for detection of entanglement resources.

For our purpose, we first characterize quantum correlations, thereby analyzing the non-n-local nature of the correlations as detected via violation of existing non-n-local inequality [16] when each of the sources generates an arbitrary two-qubit state. In this context, one may note that such a study of quantum violation was recently initiated in [7] where only two entangled sources ($n = 2$) were considered (bilocal network).

As a direct consequence of our findings in practical ground, we propose a scheme of detecting entanglement (if any) using

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networks involving independent sources where not all parties have access to random choice of inputs. Such a protocol, relying on n-local correlations generated in "fixed input" measurement scenario [16], serves the purpose of bipartite entanglement detection. Fixing measurements of some of the parties makes implementation of our protocol easier compared to the simplest standard Bell scenario of measurements. However, it must be pointed out that more easier implementation of protocols (compared to the scenario to be considered presently) may be possible if parties are allowed to randomly select from some suitable measurement settings which are more easily implementable. But, we do not consider those easily implementable measurement setting scenarios with a motivation to detect entanglement in absence of random input selections.

Recently, n-local networks involving sources distributing multipartite entanglement have been designed in [17]. However, in such a measurement scenario, all the parties had access to random choice of measurements. To verify quantum nonlocality even in absence of randomness in input selection (by some of the parties), we consider a measurement scenario where now three independent sources generate tripartite quantum states. In this context, we have designed a set of nonlinear Bell inequalities, a violation of which suffices to detect non-n-locality. The nonlinear trilocal network scenario is then used for the purpose of tripartite entanglement detection. Our protocol (characterized by fixed measurement setting by two parties) can detect both biseparable and genuine entanglement (some members of GGHZ and W classes) of pure tripartite quantum states. Interestingly, it can be used to distinguish between genuine entanglement and biseparable entanglement of pure states and can even specify the exact nature of biseparable entanglement. Finally, we conjecture generalization of our protocol for detecting entanglement of multipartite ($m \geqslant 4$) pure states. Apart from entanglement detection, the study of analyzing quantumness of network correlations may be contributory in the study of various information processing tasks such as distribution of quantum key (QKD) [20–23], generation of private randomness [24,25], Bayesian game theoretic applications [26], etc.

The rest of our work is organized as follows. We start with discussing the motivation behind our work in Sec. II followed by some basic preliminaries in Sec. III. In Sec. IV first we analyze the nature of quantum correlations generated in n-local linear network [16] using n number of bipartite quantum states followed by proposal of the scheme for bipartite entanglement detection. In Sec. V, first we derive the set of Bell inequalities for the nonlinear trilocal network scenario, then study violation of corresponding inequalities by pure tripartite quantum states, and then design tripartite entanglement detection scheme for some pure tripartite states. In Sec. VI, we generalize the nonlinear trilocal network to a nonlinear n-local network scenario when each of n independent sources generates an m-partite ($m \geqslant 4$) state. Finally, we end with some concluding remarks in Sec. VII.

II. MOTIVATION

Nonlocal behavior (Bell nonlocality) of correlations acts as a signature of presence of entanglement distributed (by a common source) among the parties who perform local measurements on their respective particles forming the entangled state. In a network scenario, specifically for a bilocal network, Gisin *et al.* proved that all bipartite entangled states violate the bilocal inequality (see Sec. III) indicating nonbilocality of corresponding network correlations [13]. Their findings [7] generate the idea of using a bilocal network to detect entanglement of the states distributed by the sources. This idea basically motivates this work. We exploit the nonclassical nature of quantum correlations generated in a network (involving n independent sources) where all the parties do not have access to random input selection. Subsequently, we use the observations for detecting entanglement of bipartite states involved in the network. We not only confine within the scope of bipartite entanglement, but consider tripartite entanglement also.

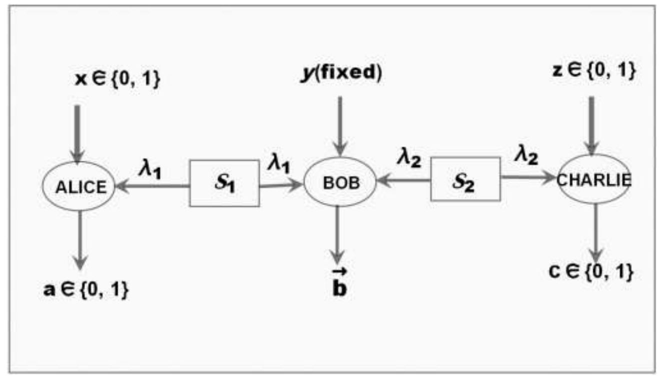

FIG. 1. Schematic diagram of bilocal network [13,14]. In the P_{14} scenario, y denotes fixed measurement of Bob together with $\vec{b}$ referring to four outputs $b_0b_1 = 00,\ 01,\ 10,\ 11$.

III. PRELIMINARIES

A. Bilocal scenario

A bilocal network (Fig. 1) was framed in [13,14]. It is a network of three parties, say, Alice (A), Bob (B), and Charlie (C), and two sources $\mathbf{S}_1$ and $\mathbf{S}_2$ arranged linearly. Sources $\mathbf{S}_1$ and $\mathbf{S}_2$ are independent to each other (*bilocal assumption*). Each of $\mathbf{S}_1$ and $\mathbf{S}_2$ sends a physical system characterized by variables λ_1 and λ_2, respectively. Intermediate party Bob gets two particles (one from each source). In P^{14} scenario [13,14], each of Alice and Charlie performs any one of two binary output measurements on their respective subsystems: $x,\ z \in \{0, 1\}$ denote respective input sets for Alice and Charlie whereas their outputs are labeled as $a,\ c \in \{0, 1\}$. Bob performs a single (fixed) measurement (y) having four outcomes: $b = \overrightarrow{\mathbf{b}} = b_0b_1 = 00, 01, 10, 11$ on the joint state of the two subsystems received from $\mathbf{S}_1$ and $\mathbf{S}_2$.

Correlations generated in the network are local if these can be decomposed as $P_{14}(a, b, c|x, y, z) = \iint d\lambda_1 d\lambda_2 \rho(\lambda_1, \lambda_2) V$

$$\text{with } V = P_{14}(a|x, \lambda_1)P_{14}(b|y, \lambda_1, \lambda_2)P_{14}(c|z, \lambda_2). \quad (1)$$

Tripartite correlations $P_{14}(a, b, c|x, y, z)$ are bilocal if these can be written in the above form [Eq. (1)] along with the constraint (referred to as *bilocal constraint*)

$$\rho(\lambda_1, \lambda_2) = \rho_1(\lambda_1)\rho_2(\lambda_2) \quad (2)$$

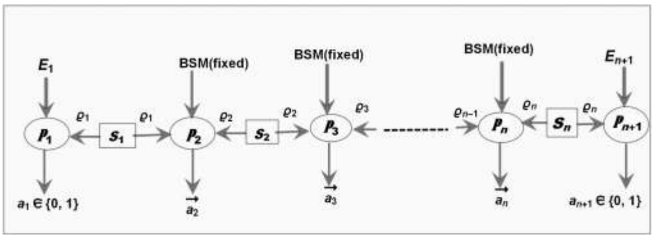

FIG. 2. Schematic diagram of quantum n-local linear network. E_1 and E_{n+1} stand for observables corresponding to binary inputs x_1 and x_{n+1} of $\mathcal{P}_1$ and $\mathcal{P}_{n+1}$, respectively. Here, each of $\mathcal{P}_i$ $(i = 2, ..., n)$ performs asingle measurement denoted by BSM, which stands for complete Bell basis measurement with $\vec{a}_i$ referring to four outputs $a_{i0}a_{i1} = 00,\ 01,\ 10,\ 11$.

imposed on the probability distributions of the hidden variables λ_1, λ_2. A linear extension of this model involving n independent sources and $n+1$ parties was made in [16].

Nonbilocality of tripartite correlations is guaranteed if these violate the inequality $\sqrt{|I|} + \sqrt{|J|} \leqslant 1$ (for details, see [13]). Quantum violation of the bilocal inequality was pointed out in [7]. Based on their findings it can be said that any bipartite two-qubit entangled state violates the bilocal inequality. Linear extension of bilocal network, referred to as *n-local linear network*, was given in [16] where the number of independent sources is n. n-local quantum linear network is considered for our purpose (see Fig. 2).

B. Complete Bell basis and GHZ basis measurement

Both of these measurements are instances of quantum entangled (joint) measurements. The operator of complete Bell basis measurement [7], often referred to as "Bell state measurement" (BSM), is represented in terms of its four eigenvectors (Bell states):

$$|\phi^{\pm}\rangle = \frac{|00\rangle \pm |11\rangle}{\sqrt{2}}, \tag{3}$$

$$|\psi^{\pm}\rangle = \frac{|01\rangle \pm |10\rangle}{\sqrt{2}}. \tag{4}$$

Analogous to the bipartite entangled measurement of BSM, the operator corresponding to tripartite entangled measurement of complete GHZ basis measurement (GSM) is given in terms of the GHZ basis [15]:

$$|\phi_{mnk}\rangle_{\text{GHZ}} = \frac{1}{\sqrt{2}} \sum_{r=0}^{1} (-1)^{m*r} |r\rangle |r \oplus n\rangle |r \oplus k\rangle,\ m, n, k \in \{0, 1\}. \tag{5}$$

IV. QUANTUM VIOLATION OF LINEAR *n*-LOCAL INEQUALITY

Here, we consider an n-local linear network [16] involving quantum states (see Fig. 2). Let each of n independent sources generate a two-qubit state: source $\mathbf{S}_i$ generating state ϱ_i $(i = 1, 2, \ldots, n)$. Two qubits of state ϱ_i are sent to parties $\mathcal{P}_i$ and $\mathcal{P}_{i+1}$ $(i = 1, 2, \ldots, n)$. The overall joint quantum system involved in the network is $\otimes_{i=1}^{n} \varrho_i$. After receiving qubits, each of the extreme two parties $\mathcal{P}_1$ and $\mathcal{P}_{n+1}$ performs projective measurements in any of two arbitrary directions locally on their respective particles: $\mathcal{P}_1$ chooses any one of directions $\vec{\alpha}_0$ and $\vec{\alpha}_1$ (say) whereas for $\mathcal{P}_{n+1}$ let the directions be along any one $\vec{\beta}_0$ and $\vec{\beta}_1$. Each of remaining $n-1$ intermediate parties $\mathcal{P}_i$ $(i = 2, \ldots, n-1)$ performs a complete Bell-basis measurement (fixed setting) on the joint state of their respective two particles received from adjoining sources $\mathbf{S}_i$ and $\mathbf{S}_{i+1}$ (see Fig. 2). $(n+1)$-partite correlations generated in the network are then used to test the n-local inequality [16]

$$\sqrt{|I_{14}|} + \sqrt{|J_{14}|} \leqslant 1. \tag{6}$$

Terms appearing in the above equation are detailed in Table I.

Clearly, excepting the extreme two parties A_1 and A_{n+1}, none of the remaining $n-1$ parties have access to random choice of measurements. Under such circumstances, we consider two separate cases.

Network involving pure states. Let each of the n sources generate a pure two-qubit state. To be precise, say $\mathbf{S}_i$ emits

$$\varrho_i = \gamma_{0i}|00\rangle + \gamma_{1i}|11\rangle, \tag{7}$$

where γ_{0i} and γ_{1i} $(i = 1, \ldots, n)$ are positive real Schmidt coefficients [27] satisfying normalization condition $\gamma_{oi}^2 + \gamma_{1i}^2 = 1$. ϱ_i is entangled for any nonzero value of both γ_{0i} and γ_{i1} $(i = 1, 2, \ldots, n)$, i.e., $\gamma_{0i}\gamma_{1i} > 0$. Maximizing over all possible projective measurement directions of extreme two parties A_1 and A_{n+1}, the upper bound of violation ($\mathcal{B}_{14}$, say) of Eq. (6) turns out to be

$$\mathcal{B}_{14} = \mathcal{B}_{14}^{(\text{pure})} = \sqrt{1 + 2^n \Pi_{i=1}^{n} \gamma_{oi}\gamma_{1i}}. \tag{8}$$

$\mathcal{B}_{14} > 1$ implies that all the pure states involved in the network are entangled. Hence, up to the existing sufficient criterion

TABLE I. Details of the terms appearing in Eq. (6). E_1 and E_{n+1} denote respective observables corresponding to binary inputs x_1 and x_{n+1} of parties $\mathcal{P}_1$ and $\mathcal{P}_{n+1}$. $a_1, a_{n+1} \in \{0, 1\}$ stand for corresponding outputs.

I_{14} and J_{14}	Correlators	Measurement and outputs of $\mathcal{P}_i$ $(i = 2, \ldots, n)$
$I_{14} = \frac{1}{4} \sum_{x_1, x_{n+1}=0,1} \langle E_1 E_2^0 \ldots E_n^0 E_{n+1} \rangle$	$\langle E_1 E_2^y \ldots E_n^y E_{n+1} \rangle = \sum_U (-1)^{a_1 + a_{2y} + \ldots + a_{ny} + a_{n+1}} V_{14}$	E_i^y:observable of $\mathcal{P}_i$
	$U = \{a_1, a_{20}, a_{21}, \ldots, a_{n0}, a_{n1}, a_{n+1}\}$	Corresponding to a single input
$J_{14} = \frac{1}{4} \sum_{x_1, x_{n+1}=0,1} (-1)^{x_1 + x_{n+1}} \langle E_1 E_2^1 \ldots E_n^1 E_{n+1} \rangle$	$V_{14} = P(a_1, \overrightarrow{a_2}, \ldots, \overrightarrow{a_n}, a_{n+1} \vert x_1, x_{n+1})$	having 4 outputs:
		$\overrightarrow{a_i} = a_{i0}a_{i1} = 00,\ 01,\ 10,\ 11$

given by Eq. (6) for detecting nonbilocality, nonbilocal correlations are generated in a network only if all pure states involved in the network are entangled.

Network involving mixed bipartite states. Let us now consider the case when each of $\mathbf{S}_j$ emits a mixed bipartite state (density matrix formalism)

$$\varrho_j = \frac{1}{2^2}\sum_{i_1,i_2=0}^{3} t^{(j)}_{i_1 i_2}\sigma^1_{i_1}\bigotimes\sigma^2_{i_2}, \tag{9}$$

where σ_0^q stands for the identity operator of the Hilbert space which is associated with qubit q and $\sigma^q_{i_q}$, denote the Pauli operators along three mutually perpendicular directions, $i_q = 1, 2, 3$. $t^{(j)}_{i_1 i_2}$ $(i, j = 1, 2, 3)$ denote the elements of the correlation tensor $T^{(j)}$ (say) of the bipartite state ϱ_j. Polar value decomposition of correlation tensor $(T^{(j)})$ for each of ϱ_j generates the matrix $U^{(j)}M^{(j)} = T^{(j)}$ where $U^{(j)}$ denotes a unitary matrix and $M^{(j)} = \sqrt{(T^{(j)})^\dagger T^{(j)}}$ having eigenvalues $\lambda_1^{(j)} \geqslant \lambda_2^{(j)} \geqslant \lambda_3^{(j)}$. The polar decomposition of $\varrho^{(j)}$ and $\varrho^{(j+1)}$ characterize the fixed measurement (BSM) of A_j $(j = 2, \ldots, n)$ who performs suitable local unitaries over subsystems received from sources $\mathbf{S}_j$ and $\mathbf{S}_{j+1}$ (for detailed discussion on the methodology used here, see [7]). The upper bound of violation $(\mathcal{B}_{14})$ now turns out to be

$$\mathcal{B}_{14} = \mathcal{B}_{14}^{(\text{mixed})} = \sqrt{\Pi_{j=1}^n \lambda_1^{(j)} + \Pi_{j=1}^n \lambda_2^{(j)}}. \tag{10}$$

Now, let none of ϱ_j $(j = 1, \ldots, n)$ violate standard Bell-CHSH inequality, i.e., by Horodecki criterion [28]

$$\mathcal{B}^{(j)}_{\text{CHSH}} = \sqrt{\left(\lambda_1^{(j)}\right)^2 + \left(\lambda_2^{(j)}\right)^2} \leqslant 1, \tag{11}$$

where $\mathcal{B}^{(j)}_{\text{CHSH}}$ denotes the upper bound of violation of Bell-CHSH inequality by ϱ_j. This in turn indicates that for each of ϱ_j $(j = 1, \ldots, n)$, $\lambda_i^{(j)}$ $(i = 1, 2, 3) < 1$. Under such circumstances, Eq. (10) gives

$$\begin{aligned}\mathcal{B}_{14} &< \max_{k\neq l}\sqrt{\lambda_1^{(k)}\lambda_1^{(l)} + \lambda_2^{(k)}\lambda_2^{(l)}}, \quad k, l = 1, \ldots, n\\ &\leqslant \sqrt{\left(\lambda_1^{(k)}\right)^2 + \left(\lambda_2^{(k)}\right)^2}\sqrt{\left(\lambda_1^{(l)}\right)^2 + \left(\lambda_2^{(l)}\right)^2}\\ &= \mathcal{B}^{(k)}_{\text{CHSH}}\mathcal{B}^{(l)}_{\text{CHSH}}\\ &\leqslant 1.\end{aligned} \tag{12}$$

Hence, $\mathcal{B}_{14} > 1$ implies that at least one of the states ϱ_j generated by $\mathbf{S}_j$ is Bell-CHSH nonlocal.

A. Bipartite entanglement detection

Let there be n unknown bipartite quantum states Φ_i generated by n distinct sources $\mathbf{S}_i$ $(i = 1, \ldots, n)$. All these n sources being spatially separated, they are independent of each other. In order to detect whether at least one of Φ_i is entangled or not, let the sources be arranged linearly and the states be distributed among $n+1$ parties $\mathcal{P}_i$ $(i = 1, \ldots, n+1)$ so as to form a n-local network (Fig. 2). Let each of $\mathcal{P}_1$ and $\mathcal{P}_{n+1}$ perform projective measurements in any one of two arbitrary directions whereas intermediate $n-1$ parties (receiving two particles each) perform complete Bell basis measurement (BSM). Practical implementation of this protocol, where only some of the parties $(\mathcal{P}_1, \mathcal{P}_{n+1})$ have to choose randomly from a set of two measurements, is easier compared to any protocol where none of the parties involved perform fixed measurement. $(n+1)$-partite correlations generated therein are used to test the n-local inequality [Eq. (6)]. Observation of violation of the inequality guarantees that at least one of Φ_i is entangled. Utility of the violation of Eq. (6) is already justified in the previous subsection. Clearly, this protocol detects entanglement of all the states involved in a device-dependent manner (as each of the intermediate parties have to perform BSM thereby making the scheme depending on inner working of the device) in case all Φ_i $(i = 1, \ldots, n)$ are identical copies of an unknown quantum state.

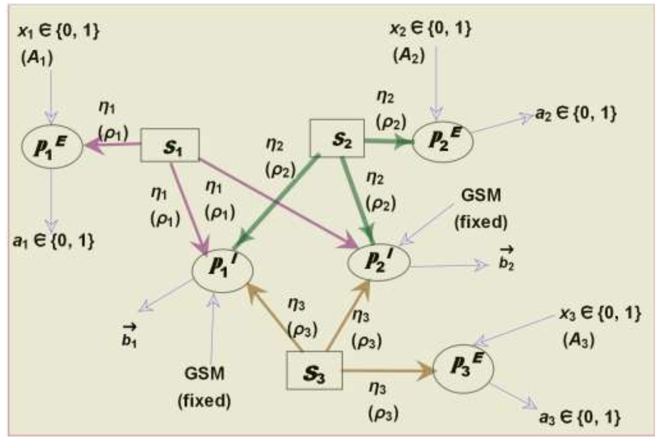

FIG. 3. Trilocal nonlinear network. Source $\mathbf{S}_i$ is characterized by hidden variable η_i $(i = 1, 2, 3)$. In case of quantum network $\mathbf{S}_i$ generates tripartite quantum state ρ_i. A_i denotes observables corresponding to binary inputs x_i of $\mathcal{P}_i$ $(= 1, 2, 3)$, respectively. Here, each of $\mathcal{P}_1^I$, $\mathcal{P}_2^I$ performs single measurement (GHZ basis measurement) denoted by "GSM" with three-dimensional vector $\vec{b}_i$ now referring to eight outputs $b_{i0}b_{i1}b_{i2}$= 000, 001, 010, 100, 101, 110, 011, 111.

Having used a n-local linear network for the purpose of bipartite entanglement detection, we now proceed to do the same for some families of pure tripartite entangled states. For that, we first analyze trilocal nonlinear network scenario.

V. TRILOCAL NONLINEAR NETWORK SCENARIO

The scenario is based on a five-party $(\mathcal{P}_1^E, \mathcal{P}_2^E, \mathcal{P}_3^E, \mathcal{P}_1^I, \mathcal{P}_2^I)$ network involving three independent sources $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$ (see Fig. 3). Source $\mathbf{S}_i$ is characterized by hidden variable η_i $(i = 1, 2, 3)$. Source independence implies existence of independent probability distributions

$$\Lambda(\eta_1, \eta_2, \eta_3) = \Lambda_1(\eta_1)\Lambda_2(\eta_2)\Lambda_3(\eta_3), \tag{13}$$

where $\int d\eta_i \Lambda_i(\eta_i) = 1\ \forall\ i$. Source $\mathbf{S}_i$ sends particles to parties $\mathcal{P}_1^I$, $\mathcal{P}_2^I$, $\mathcal{P}_i^E$ $(i = 1, 2, 3)$. Parties $\mathcal{P}_1^I$ and $\mathcal{P}_2^I$ receiving three particles (one from each source) are referred to as *intermediate* parties and the remaining three parties $\mathcal{P}_1^E, \mathcal{P}_2^E, \mathcal{P}_3^E$, each receiving a single particle, are referred to as *extreme* parties. Let x_1, x_2, $x_3(\in \{0, 1\})$ stand for binary inputs of parties $\mathcal{P}_1^E$, $\mathcal{P}_2^E$, $\mathcal{P}_3^E$, respectively, whereas a_1, a_2, $a_3(\in \{0, 1\})$ correspond to the respective outputs. Each of $\mathcal{P}_1^I$ and $\mathcal{P}_2^I$ has access to single input giving eight

TABLE II. Detailing of the terms used in Eq. (15). A_i denote the observable for input x_i of $\mathcal{P}_i^E$ $(i = 1, 2, 3)$ whereas B_1, B_2 denote observable corresponding to single input of $\mathcal{P}_1^I$ and $\mathcal{P}_2^I$, respectively.

Correlators
$I^{(18)}_{m_1(n_1),m_2(n_2),i} = \frac{1}{8}\sum_{x_1,x_2,x_3=0,1}(-1)^{i*(x_1+x_2+x_3)}\langle A_{1,x_1}A_2^{m_1(n_1)}A_3^{m_2(n_2)}A_{4,x_2}A_{5,x_3}\rangle,\ i,\ m_1,\ m_2,\ n_1,\ n_2\ \in \{0,1\}$
$\langle A_{1,x_1}B_1^{m_1(n_1)}B_2^{m_2(n_2)}A_{2,x_2}A_{3,x_3}\rangle = \sum_{\mathcal{C}}(-1)^h P_{18}(a_1,\vec{b_1},\vec{b_2},a_2,a_3\vert x_1,x_2,x_3)$
where $\mathcal{C} = \{a_1,a_2,a_3,b_{10},b_{11},b_{12},b_{20},b_{21},b_{22}\}$ and $h = a_1+a_2+a_3+s_{m_1(n_1)}(b_{10},b_{11},b_{12})+s_{m_2(n_2)}(b_{20},b_{21},b_{22})$
with functions $s_i(x,y,z)$ being defined as $s_0(x,y,z) = x+y+z+1$ and $s_1(x,y,z) = x*y+y*z+x*z$

outputs $\vec{b_1} = (b_{10}, b_{11}, b_{12})$ and $\vec{b_2} = (b_{20}, b_{21}, b_{22})$ ($b_{ij} \in \{0,1\}\ \forall\ i = 1,2$ and $j \in \{0,1,2\}$) denote the outputs of parties $\mathcal{P}_1^I$ and $\mathcal{P}_2^I$, respectively. Parties are not allowed to communicate between themselves. Correlations generated in this network scenario are *trilocal* if they can be factorized as follows:

$$\begin{aligned}
&P_{18}(a_1,\vec{b_1},\vec{b_2},a_2,a_3\vert x_1,x_2,x_3)\\
&\quad = \iiint d\eta_1 d\eta_2 d\eta_3 \Lambda(\eta_1,\eta_2,\eta_3)W,\\
\text{where } W &= P_{18}(a_1\vert x_1,\eta_1)P_{18}(\vec{b_1}\vert\eta_1,\eta_2,\eta_3)P_{18}(\vec{b_2}\vert\eta_1,\eta_2,\eta_3)\\
&\quad \times P_{18}(a_2\vert x_2,\eta_2)P_{18}(a_3\vert x_3,\eta_3) \qquad (14)
\end{aligned}$$

along with the restriction imposed by Eq. (13). Under source independence restriction [Eq. (13)], correlations which cannot be decomposed as above [Eq. (23)] are said to be *nontrilocal* in nature. It may be noted that the network scenario introduced here is in some extent similar to that of the scenario discussed in [17] where each of the parties involved has the freedom to choose from a set of two measurements. So, the scenario in the present discussion and that introduced in [17] differ on the basis of whether the intermediate parties perform a single measurement or not. Correspondingly, the correlations characterizing the measurement scenarios and the inequalities involved therein are different from those discussed in [17]. We now derive a set of sufficient criteria in the form of nonlinear Bell-type inequalities sufficient to detect nontrilocal correlations.

Theorem 1. For any trilocal five-partite correlation, each of the following inequalities necessarily holds:

$$\sqrt[3]{\left|I^{(18)}_{m_1,m_2,0}\right|} + \sqrt[3]{\left|I^{(18)}_{n_1,n_2,1}\right|} \leqslant 1\,\forall\, m_1,\ m_2,\ n_1,\ n_2\ \in \{0,1\}. \qquad (15)$$

For details of the correlators used in Eq. (15), see Table II.

Proof. For proof, see Appendix A.

The set of 16 inequalities given by Eq. (15) being only necessary criteria of trilocality, there may exist nontrilocal correlations satisfying all of them. However, violation of at least one of these inequalities guarantees nontrilocality of the correlations. Violation of Eq. (15) for at least one possible (m_1, m_2, n_1, n_2) is thus sufficient for detecting nontrilocality of corresponding correlations.

A. Quantum violation

Consider a network involving three independent sources $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$, each generating a three-qubit state $\rho^{(i)}$ (see Fig. 3). The overall quantum state involved in the network becomes

$$\rho_{12345} = \rho^{(1)}\otimes\rho^{(2)}\otimes\rho^{(3)}. \qquad (16)$$

After the qubits are distributed from the sources, no communication takes place between the parties who now perform measurements on their respective subsystems. Each of $\mathcal{P}_1^I$ and $\mathcal{P}_2^I$ performs complete GHZ basis measurement (GSM) on the joint state of the three qubits that each of them receives from the three sources. Each of $\mathcal{P}_1^E$, $\mathcal{P}_2^E$, and $\mathcal{P}_3^E$ performs projective measurements on their single qubit in any of two arbitrary directions: $\mathcal{P}_i^I$ $(i = 1, 2, 3)$ measures in any one of $\vec{\gamma}_{i0}$ and $\vec{\gamma}_{i1}$ directions.

Interestingly, if each of the sources $\mathbf{S}_i$ generates arbitrary tripartite product state

$$\rho_i = \otimes_{j=1}^3(v_{0ij}\vert 0\rangle + v_{1ij}\vert 1\rangle)(\vert v_{0ij}\vert^2 + \vert v_{1ij}\vert^2 = 1), \qquad (17)$$

none of the inequalities given by Eq. (15) are violated. We now proceed to discuss some possible cases of quantum violation of inequalities given by Eq. (15). For our purpose, we consider tripartite pure states.

Let each of the sources generate an arbitrary biseparable (in 12/3 cut) entangled state

$$\left|\varphi^i_{(12/3)}\right\rangle = (c_{0i}\vert 00\rangle_{12} + c_{1i}\vert 11\rangle_{12}) \otimes (v_{0i}\vert 0\rangle_3 + v_{1i}\vert 1\rangle_3) \qquad (18)$$

with $v_{0i}^2 + v_{1i}^2 = 1$ and $c_{0i}^2 + c_{1i}^2 = 1$ (v_{ij}, c_{ij} are the Schmidt coefficients) [27]. Now, compatible with the arrangement of the sources and parties in this network, let the first qubit of each $\rho_i = \vert\varphi^i_{(12\vert 3)}\rangle\langle\varphi^i_{(12\vert 3)}\vert$ $(i = 1, 2, 3)$ is sent to the extreme parties: $\mathcal{P}_1^E$, $\mathcal{P}_2^E$, and $\mathcal{P}_3^E$ receiving the first qubit of ρ_1, ρ_2, and ρ_3, respectively, whereas the second and third qubits of each ρ_i are sent to the intermediate parties: $\mathcal{P}_1^I$ receives the second qubit of ρ_1, ρ_2, ρ_3 and $\mathcal{P}_3^I$ receives the third qubit of these states. Violation of Eq. (15) is observed for some members of this family [Eq. (18)]. Violation is also observed if each of $\mathbf{S}_i$ generates some states having biseparable entanglement in 13/2 cut:

$$\left|\varphi^i_{(13/2)}\right\rangle = (c_{0i}\vert 00\rangle_{13} + c_{1i}\vert 11\rangle_{13}) \otimes (v_{0i}\vert 0\rangle_2 + v_{1i}\vert 1\rangle_2). \qquad (19)$$

However, violation is impossible if $\mathbf{S}_i$ generates any member from the family of biseparable entangled states having entanglement among its second and third qubits:

$$\left|\varphi^i_{(23/1)}\right\rangle = (c_{0i}\vert 00\rangle_{23} + c_{1i}\vert 11\rangle_{23}) \otimes (v_{0i}\vert 0\rangle_1 + v_{1i}\vert 1\rangle_1). \qquad (20)$$

At this junction, it should be noted that violation of Eq. (15) depends on the order of distribution of qubits of each ρ_i ($i = 1, 2, 3$) among the parties. Compatible with the network scenario (Fig. 3), when first qubit of each ρ_i $(i = 1, 2, 3)$ is sent to the extreme parties and remaining two qubits of each ρ_i are

TABLE III. Exploring some specific instances of nontrilocal nature of correlations observed when some tripartite pure quantum states are used in the nonlinear trilocal network under the assumption that each of $\mathcal{P}_1^E$, $\mathcal{P}_2^E$, and $\mathcal{P}_3^E$ receives the first qubit of ρ_1, ρ_2, and ρ_3, respectively, whereas the remaining two qubits of each ρ_i are received by the intermediate parties. No violation of trilocal inequalities (at least one) is, however, obtained when tripartite state has entangled second and third qubits [Eq. (20)] as $\mathcal{B}_{18} \leqslant 1$.

State (ρ_i generated by $\mathbf{S}_i$)	State parameters giving violation
$\lvert\varphi_{\mathrm{GHZ}}^{(i)}\rangle$ [Eq. (21)]	$\beta_1 = 0.72, \beta_2 = 0.75, \beta_3 = 0.7$
$\lvert\varphi_{W}^{(i)}\rangle$ [Eq. (22)]	$\omega_{1i} = 0.558327, \omega_{2i} = 1.5708, \ \forall\, i \in \{1, 2, 3\}$
$\lvert\varphi_{(12\lvert 3)}^{i}\rangle$ [Eq.(18)]	$c_{0i} = 0.592368, \ v_{0i} = 1, \ \forall\, i \in \{1, 2, 3\}$
$\lvert\varphi_{(13\lvert 2)}^{i}\rangle$ [Eq. (19)]	$c_{0i} = 1.5708 - \imath\, 0.15776, \ v_{0i} = 1, \ \forall\, i \in \{1, 2, 3\}$
$\lvert\varphi_{(23\lvert 1)}^{i}\rangle$ [Eq.(20)]	No violation is obtained. Upper bound ($\mathcal{B}_{18}$, say) of trilocal inequalities [Eq. (15)] for identical copies $\mathcal{B}_{18} = \mathrm{Max}[2^{\frac{2}{3}}\lvert c_{01}c_{11}\rvert, (c_{01}^4 + 4c_{01}^3c_{11}^3 + c_{11}^4)^{\frac{1}{3}}]$ where $c_{k1} = c_{k2} = c_{k3}, \ k = 0, 1$

received by the intermediate parties (as discussed), violation is observed in networks involving biseparable entanglement in 13/2 [Eq. (19)] or 12/3 [Eq. (18)] cuts only. But, violation is not observed if ρ_i have biseparable entanglement in 23/1 cut [Eq. (20)]. But, networks involving biseparable entanglement in 23/1 [Eq. (20)] cut also gives violation if the second qubit of each ρ_i ($i = 1, 2, 3$) is sent to the extreme parties and the remaining two qubits of each ρ_i are received by the intermediate parties. However, violation of any one of the trilocal inequalities given by Eq. (15) is not always arrangement (of qubits) specific. We consider genuine entanglement in support of our claim.

Let each of $\mathbf{S}_i$ in the nonlinear trilocal network now generate a generalized GHZ (GGHZ) state ([29]) $\rho_i = \lvert\varphi_{\mathrm{GHZ}}^{(i)}\rangle\langle\varphi_{\mathrm{GHZ}}^{(i)}\rvert$ where

$$\left|\varphi_{\mathrm{GHZ}}^{(i)}\right\rangle = \cos(\beta_i)|000\rangle + \sin(\beta_i)|111\rangle, \quad \beta_i \in \left[0, \frac{\pi}{4}\right]. \tag{21}$$

Contrary to biseparable entanglement, nontrilocal correlations are obtained in the network (see Table. III) for some states from the GGHZ family [Eq. (21)] irrespective of distribution of qubits of each of the states (ρ_i). Analogous observation is obtained when W states [30] are involved in the network:

$$\begin{aligned}\left|\varphi_{W}^{(i)}\right\rangle &= \cos\omega_{2i}\sin\omega_{1i}|001\rangle + \sin\omega_{2i}\sin\omega_{1i}|010\rangle \\ &\quad + \cos\omega_{1i}|100\rangle,\ \omega_{1i}, \quad \omega_{2i} \in \left[0, \frac{\pi}{4}\right].\end{aligned} \tag{22}$$

Now, if both biseparable and genuine entanglement of the W state [Eq. (22)] are used in the network, violation again depends on arrangement of qubits. Here, it should be pointed out that if one of the three tripartite pure states generated by the sources is a product state, then violation of trilocal inequalities cannot be observed even if the remaining two states are entangled.

Based on the above analysis of quantum violation and the fact that such violation is sufficient to detect nontrilocal nature of network correlations, we now design a scheme to detect both biseparable and genuine entanglement of tripartite pure states. But, it may be pointed out that this scheme may fail to detect presence of entanglement in some cases as violation is not possible for all tripartite pure entangled states.

B. Tripartite pure entanglement detection

Consider a nonlinear trilocal network. The three unknown pure tripartite states κ_1, κ_2, and κ_3 are generated by $\mathbf{S}_1$, $\mathbf{S}_2$, and $\mathbf{S}_3$ in the network. Distribution of qubits among the parties plays a significant role in violation of trilocal inequalities. Consequently, for designing a scheme of entanglement detection, we consider all the possible arrangement of qubits. The protocol breaks up into 27 phases: $t_{i,j,k}$ ($i, j, k \in \{1, 2, 3\}$). In phase $t_{i,j,k}$, for every possible value of $i, j, k \in \{1, 2, 3\}$, ith, jth, kth qubits of $\kappa_1, \kappa_2, \kappa_3$, respectively, are sent to extreme parties. Hence, $\mathcal{P}_1^E$, $\mathcal{P}_2^E$, and $\mathcal{P}_3^E$ receive ith, jth, kth qubits of $\kappa_1, \kappa_2, \kappa_3$, respectively (for more details see Table VI in Appendix B). The remaining qubits of each of the unknown states are distributed among the intermediate parties in any pattern compatible with the nonlinear trilocal network scenario (Fig. 3). One may note that ordering of the phases is not essential. After receiving the particles, in each of these phases, the parties perform measurements on their respective subsystems. Correlated statistics are then used to test the trilocal inequalities [Eq. (15)]. If violation of at least one of the inequalities is observed in at least one phase, then each of κ_1, κ_2, and κ_3 is a tripartite entangled state whereas violation in all the phases ensures genuine entanglement of all the three unknown states. In the protocol, either violation occurs in no phase or in specific number of phases (see Table IV).

Interestingly, comparison of the possible nature of biseparable entanglement of $\kappa_1, \kappa_2, \kappa_3$ from Table VI ensures the nature of entanglement of each of the unknown states. To be more precise, at the end of the protocol, one can detect which of the three unknown states is genuinely entangled and which one is biseparable. Also, the specific nature of biseparable entanglement can be detected.

As already discussed, the total count of phases in which violation may be encountered is not arbitrary (see Table IV). Leaving aside the implications in the last two cases

TABLE IV. Total count of phases for which violation can be observed in the protocol is enlisted here. Implications are obvious from observations discussed in Sec. V A. In case of no violation in any of the phases, the protocol fails to detect entanglement.

Total number of phases	Implication
0	No definite conclusion
8	All are biseparable
12	Any two of the unknown states are biseparable and the remaining is genuinely entangled other than GGHZ [Eq. (21)] or W [Eq. (22)] classes
18	Only one of three unknown states is biseparable with the other two being genuinely entangled but does not belong to GGHZ or W families
$\geqslant$19 but $<$27	Each κ_i has genuine entanglement but is neither a member of GGHZ family [Eq. (21)] nor W state [Eq. (22)]
27	Each of κ_i is genuinely entangled

(corresponding to last two rows of Table IV), let us consider the remaining cases individually:

(i) Let violation be obtained in 18 phases. Then, definitely two of three unknown states are genuinely entangled but are neither a GGHZ nor W state and the remaining one is a biseparable entangled state. For instance, violation in only first 18 phases of the protocol $(t_{1,j,k}, t_{2,j,k}, j, k \in \{1, 2, 3\})$ ensures that only κ_1 is a biseparable entangled state having entanglement in 12/3 cut. This implication is obvious if one notes that the 12/3 cut biseparable entanglement is the only possible nature of entanglement of κ_1 if violation is obtained in first 18 phases (Table VI).

(ii) Violation in only 12 phases ensures that two of three unknown states are biseparable entangled and other one is genuinely entangled (other than GGHZ or W state). Consider a specific instance. Let violation be obtained in $t_{1,2,k}, t_{3,2,k}, t_{1,3,k}, t_{3,3,k},\ \forall\, k \in \{1, 2, 3\}$. Then, κ_1, κ_2 are biseparable entangled states in 13/2 and 23/1 cuts, respectively, and κ_3 is genuinely entangled.

(iii) Violation in only 8 phases ensures that all three unknown states are biseparable entangled. The nature of biseparable entanglement of each κ_i is also detected. Consider the instance where violation is obtained in phases $t_{1,2,k}, t_{3,2,k}, t_{1,3,k}, t_{3,3,k},\ \forall\, k \in \{1, 2\}$. Then, $\kappa_1, \kappa_2, \kappa_3$ are biseparable entangled states in 13/2, 23/1, and 12/3 cuts, respectively.

All these implications are direct consequences of the fact that violation of trilocal inequalities is not distribution (of qubits) specific in networks involving only genuine entanglement of GGHZ or W states whereas the same is crucial if at least one of the sources generates biseparable entanglement or genuine entanglement other than GGHZ [Eq. (21)] and W [Eq. (22)] classes.

VI. *n*-LOCAL NONLINEAR NETWORK SCENARIO

A trilocal nonlinear network can be extended to a network involving $2n-1$ parties and n independent sources, each generating an n-partite state. Each of n number of parties $\mathcal{P}_i^E$ $(i = 1, 2, \ldots, n)$ (say) receives only one particle and are referred to as *extreme* parties whereas each of remaining $n-1$ parties $\mathcal{P}_i^I$ $(i = 1, 2, \ldots, n-1)$, referred to as *intermediate* party, receives n particles (each from one source). Let $x_i \in \{0, 1\}$ and $a_i \in \{0, 1\}$ denote the binary input and output, respectively, of $\mathcal{A}_i$ $(i = 1, 2, \ldots, n)$. Each of $\mathcal{B}_i$ $(i = 1, 2, \ldots, n-1)$ performs a fixed measurement having 2^n outputs labeled as a n-dimensional vector $\vec{b}_i = (b_{i0}, \ldots, b_{in-1})$.

After receiving qubits from the sources, parties do not communicate. $(2n-1)$-partite correlations are n-local if they can be decomposed as

$$P_{12^n}(a_1, \vec{b}_1, \ldots, \vec{b}_{n-1}, a_2, \ldots, a_n | x_1, \ldots, x_n) = \int \ldots \int d\eta_1 \ldots d\eta_n \Lambda(\eta_1, \ldots, \eta_n) W_n$$
$$\text{where } W_n = \Pi_{i=1}^{n} P_{12^n}(a_i|x_i, \eta_i) \Pi_{i=1}^{n-1} P_{12^n}(\vec{b}_i|\eta_1, \ldots, \eta_n) \quad (23)$$

together with the constraint

$$\Lambda(\eta_1, \eta_2, \ldots, \eta_n) = \Pi_{i=1}^{n} \Lambda_i(\eta_i), \quad (24)$$

where η_i characterizes source $\mathbf{S}_i$ and $\int d\eta_i \Lambda_i(\eta_i) = 1\ \forall\, i \in \{1, \ldots, n\}$. Correlations inexplicable in the above form are non-n-local. The n-local inequalities are given by the following theorem.

Theorem 2. Any n-local $(2n-1)$-partite correlation term necessarily satisfies

$$\sqrt[n]{\left|I_{f_1,\ldots,f_{n-1},0}^{(12^n)}\right|} + \sqrt[n]{\left|I_{g_1,\ldots,g_{n-1},1}^{(12^n)}\right|} \leqslant 1 \quad (25)$$

where $f_1, \ldots, f_{n-1}, g_1, \ldots, g_{n-1} \in \{0, 1\}$.

Correlators used in Eq. (25) are detailed in Table V.

TABLE V. Detailing of the terms used in Eq. (25).

Correlators related to n-local nonlinear inequalities [Eq. (25)]
$I_{f_1(g_1),\ldots,f_{n-1}(g_{n-1}),i}^{(12^n)} = \frac{1}{2^n} \sum_{x_1,\ldots,x_n=0,1} (-1)^{i*(x_1+\ldots+x_n)} \langle A_{1,x_1} B_1^{f_1(g_1)} \ldots B_{n-1}^{f_{n-1}(g_{n-1})} \ldots A_{n,x_n} \rangle$, with $i,\ f_1, \ldots,\ f_{n-1},\ g_1, \ldots,\ g_{n-1} \in \{0, 1\}$
$\langle A_{1,x_1} B_1^{f_1(g_1)} \ldots B_{n-1}^{f_{n-1}(g_{n-1})} \ldots A_{n,x_n} \rangle = \sum_{\mathcal{Y}} (-1)^h P_{12^n}(a_1, \vec{b}_1, \ldots, \vec{b}_{n-1}, \ldots, a_n | x_1, \ldots, x_n)$ where $\mathcal{Y} = \{a_1, \ldots, a_n, b_{10}, \ldots, b_{1n-1}, \ldots, b_{n-10}, \ldots, b_{n-1n-1}\}$ and $h = a_1 + \ldots + a_n + s_{f_1(g_1)}(b_{20}, \ldots, b_{2n-1}) + \ldots + s_{f_{n-1}(g_{n-1})}(b_{n-10}, \ldots, b_{n-1n-1})$ with functions $s_{i-1}(k_1, \ldots, k_n)$ being defined as the sum of all possible product terms of $k_1, \ldots, k_n$ taking i k_j's at a time $(i = 1, \ldots, n-1)$.

Proof. The proof is based on the same technique as adopted for proving Theorem 1. As mentioned in Appendix A, for proving Theorem 1 we need to relate the correlators (used in present scenario) with that introduced for designing another trilocal network scenario in [17]. Analogously, Theorem 2 can be proved following the same line of argument (as that in Theorem 1). For that one should relate correlators (Table V) introduced for the n-local nonlinear scenario here with that of n-local network developed in [17]. ■

Violation of inequalities [Eq. (25)] for at least one possible $(f_1, \dots, f_{n-1}, g_1, \dots, g_{n-1})$ ensures non n-locality of corresponding correlations.

In the quantum scenario, let each source generate an n-qubit state. Each of the intermediate parties $\mathcal{P}_1^I, \dots, \mathcal{P}_{n-1}^I$ performs complete n-dimensional GHZ basis measurement on the joint of n qubits (jth qubit coming from $\mathbf{S}_i$) whereas each of the extreme $\mathcal{P}_i^E$ $(i = 1, \dots, n)$ performs projective measurement on its respective qubit. We conjecture that quantum violation of Eq. (25) can be obtained. In support of our conjecture we provide a numerical observation for $n = 4, 5$.

Let each of n independent sources $\mathbf{S}_i$ generate n-dimensional GHZ state

$$\vartheta_n = \frac{|0, 0, \dots, 0\rangle + |1, 1, \dots, 1\rangle}{\sqrt{2}}. \tag{26}$$

Violation of at least one n-local inequality [Eq. (25)] is obtained. This ensures generation of non-n-local correlations are generated in the network for $n = 4, 5$.

VII. DISCUSSIONS

In the recent past, nonlocality of quantum network correlations under circumstances that some of the parties perform a fixed measurement has been studied extensively. The topic of our paper evolves in this direction. We analyze the nonlocal feature of quantum correlations in networks involving uncorrelated sources when some of the parties do not have the freedom to choose their inputs randomly. Deriving quantum bounds of preexisting [16] n-local inequalities [Eq. (6)] turned

TABLE VI. Detailed distribution of qubits among the extreme parties in phases of the protocol. $\forall\ j, k \in \{1, 2, 3\}$, Q_j^k denotes the kth qubit of κ_j. For each $i = 1, 2, 3$, $(i+4)^{\text{th}}$ column of the table denotes the possible nature of entanglement of unknown state κ_i other than genuine entanglement when violation of at least one trilocal inequality is obtained in the corresponding phase.

Phase	$\mathcal{P}_1^E$	$\mathcal{P}_2^E$	$\mathcal{P}_3^E$	κ_1	κ_2	κ_3
$t_{1,1,1}$	$Q_1^{(1)}$	$Q_2^{(1)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 13/2 cut	12/3 or 13/2 cut
$t_{1,1,2}$	$Q_1^{(1)}$	$Q_2^{(1)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 13/2 cut	12/3 or 23/1 cut
$t_{1,1,3}$	$Q_1^{(1)}$	$Q_2^{(1)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 13/2 cut	23/1 or 13/2 cut
$t_{1,2,1}$	$Q_1^{(1)}$	$Q_2^{(2)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 23/1 cut	12/3 or 13/2 cut
$t_{1,2,2}$	$Q_1^{(1)}$	$Q_2^{(2)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 23/1 cut	12/3 or 23/1 cut
$t_{1,2,3}$	$Q_1^{(1)}$	$Q_2^{(2)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 23/1 cut	23/1 or 13/2 cut
$t_{1,3,1}$	$Q_1^{(1)}$	$Q_2^{(3)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	23/1 or 13/2 cut	12/3 or 13/2 cut
$t_{1,3,2}$	$Q_1^{(1)}$	$Q_2^{(3)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	23/1 or 13/2 cut	12/3 or 23/1 cut
$t_{1,3,3}$	$Q_1^{(1)}$	$Q_2^{(3)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	23/1 or 13/2 cut	23/1 or 13/2 cut
$t_{2,1,1}$	$Q_1^{(2)}$	$Q_2^{(1)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 13/2 cut	12/3 or 13/2 cut
$t_{2,1,2}$	$Q_1^{(2)}$	$Q_2^{(1)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 13/2 cut	12/3 or 23/1 cut
$t_{2,1,3}$	$Q_1^{(2)}$	$Q_2^{(1)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 13/2 cut	23/1 or 13/2 cut
$t_{2,2,1}$	$Q_1^{(2)}$	$Q_2^{(2)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 23/1 cut	12/3 or 13/2 cut
$t_{2,2,2}$	$Q_1^{(2)}$	$Q_2^{(2)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 23/1 cut	12/3 or 23/1 cut
$t_{2,2,3}$	$Q_1^{(2)}$	$Q_2^{(2)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 23/1 cut	23/1 or 13/2 cut
$t_{2,3,1}$	$Q_1^{(2)}$	$Q_2^{(3)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	23/1 or 13/2 cut	12/3 or 13/2 cut
$t_{2,3,2}$	$Q_1^{(2)}$	$Q_2^{(3)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	23/1 or 13/2 cut	12/3 or 23/1 cut
$t_{2,3,3}$	$Q_1^{(2)}$	$Q_2^{(3)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	23/1 or 13/2 cut	23/1 or 13/2 cut
$t_{3,1,1}$	$Q_1^{(3)}$	$Q_2^{(1)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 13/2 cut	12/3 or 13/2 cut
$t_{3,1,2}$	$Q_1^{(3)}$	$Q_2^{(1)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 13/2 cut	12/3 or 23/1 cut
$t_{3,1,3}$	$Q_1^{(3)}$	$Q_2^{(1)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 13/2 cut	23/1 or 13/2 cut
$t_{3,2,1}$	$Q_1^{(3)}$	$Q_2^{(2)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 23/1 cut	12/3 or 13/2 cut
$t_{3,2,2}$	$Q_1^{(3)}$	$Q_2^{(2)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 23/1 cut	12/3 or 23/1 cut
$t_{3,2,3}$	$Q_1^{(3)}$	$Q_2^{(2)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	12/3 or 23/1 cut	23/1 or 13/2 cut
$t_{3,3,1}$	$Q_1^{(3)}$	$Q_2^{(3)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	23/1 or 13/2 cut	12/3 or 13/2 cut
$t_{3,3,2}$	$Q_1^{(3)}$	$Q_2^{(3)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	23/1 or 13/2 cut	12/3 or 23/1 cut
$t_{3,3,3}$	$Q_1^{(3)}$	$Q_2^{(3)}$	$Q_3^{(1)}$	12/3 or 13/2 cut	23/1 or 13/2 cut	23/1 or 13/2 cut

out to be useful for designing a protocol capable of detecting bipartite resource of entanglement distributed in the network.

Analyzing network scenarios involving bipartite entanglement sources, we have then designed networks where sources now generate tripartite quantum states. In this context, we have framed a set of trilocal inequalities [Eq. (15)], violation of which (at least one) is sufficient to guarantee nontrilocality of corresponding correlations. Discussions in Sec. V ensure that randomness in choice of inputs for every party involved is not necessary to generate nonlocal (in sense of nontrilocality) correlations even when tripartite entanglement resources are distributed in the network. Based on numerical evidence we conjecture the same for exploiting non-n-locality ($n \geqslant 4$) also. Consequently, even when all the observers cannot randomly select their respective inputs in network scenarios involving m-partite ($m \geqslant 4$) entanglement (generated by sources), nonlocal (non-n-local) correlations can be obtained.

Apart from theoretical perspectives, these trilocal network scenarios turned out to be useful on practical grounds for detection of tripartite entanglement of pure states. More interestingly, protocols designed here can discriminate between some genuinely entangled states and biseparable entanglement existing in any possible grouping of two qubits constituting the three-qubit state. In this context, it will be interesting to enhance the capability of this protocol to discriminate between arbitrary genuine entanglement and biseparable entanglement of any tripartite state. n-local nonlinear network scenario introduced here may be explored further with an objective to detect entanglement of m-partite ($m \geqslant 4$) states and also to discriminate between genuine entanglement from any other form of m-partite entanglement.

APPENDIX A

In [17], another trilocal network scenario was introduced where each of the five parties, involved in the network, performs one of two dichotomic measurements, i.e., unlike the measurement scenario introduced here, none of the parties has fixed input (for details, see [17]). Correlations generated in such a network [17] are trilocal if they satisfy

$$\sqrt[3]{|\mathcal{I}_{u_1,u_2,0}|}+\sqrt[3]{|\mathcal{I}_{v_1,v_2,1}|}\leqslant 1\ \forall\, u_1,\, u_2,\, v_1,\, v_2\, \in\{0,1\}\text{ with} \tag{A1}$$

$$\mathcal{I}_{u_1(v_1),u_2(v_2),t}=\frac{1}{8}\sum_{x_1,x_2,x_3=0,1}(-1)^{t*q}\langle\mathcal{A}_{1,x_1}\mathcal{B}_{1,y_1=u_1(v_1)}\mathcal{B}_{2,x_2=u_2(v_2)}\mathcal{A}_{2,x_2}\mathcal{A}_{3,x_3}\rangle,\quad t\in\{0,1\},\quad q=x_1+x_2+x_3, \tag{A2}$$

where

$$\langle\mathcal{A}_{1,x_1}\mathcal{B}_{1,y_1}\mathcal{B}_{2,y_2}\mathcal{A}_{2,x_2}\mathcal{A}_{3,x_3}\rangle=\sum_{a_1,b_1,b_2,a_2,a_3}(-1)^m P(a_1,b_1,b_2,a_2,a_3|x_1,y_1,y_2,x_2,x_3),\quad\text{with } m=a_1+b_1+b_2+a_2+a_3, \tag{A3}$$

where $x_i\in\{0,1\}$ denote the input whereas $a_i\in\{0,1\}$ denote the corresponding output of extreme party $\mathcal{P}_i^E$ ($i=1,2,3$). Similarly, y_1, y_2 denote input and b_1, b_2 denote output of intermediate party $\mathcal{P}_1^I, \mathcal{P}_2^I$, respectively. $\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{B}_1, \mathcal{B}_2$ denote the corresponding observables. We now proceed to prove Theorem 1.

Proof. For simplicity, we use the notations $s_{y_i}(b_{i0}, b_{i1}, b_{i2}) = s_{y_i}$ ($i = 1, 2$). Now comparison of the correlation terms related to these two scenarios gives

$$\begin{aligned}P(a_1,b_1,b_2,a_2,a_3|x_1,y_1,y_2,x_2,x_3)&=P_{18}(a_1,s_{y_1}=b_1,s_{y_2}=b_2,a_2,a_3|x_1,x_2,x_3)\\&=\sum_{\mathcal{D}}\delta_{b_1,s_{y_1}}\delta_{b_2,s_{y_2}}P_{18}(a_1,b_{10},b_{11},b_{12},b_{20},b_{21},b_{22},a_2,a_3|x_1,x_2,x_3),\end{aligned} \tag{A4}$$

where $\mathcal{D}=\{b_{10},b_{11},b_{12},b_{20},b_{21},b_{22}\}$. By Eq. (A3),

$$\begin{aligned}\langle\mathcal{A}_{1,x_1}\mathcal{B}_{1,y_1}\mathcal{B}_{2,y_2}\mathcal{A}_{2,x_2}\mathcal{A}_{3,x_3}\rangle=\sum_{a_1,a_2,a_3}(-1)^{a_1+a_2+a_3}[&P(a_1,0,0,a_2,a_3|x_1,y_1,y_2,x_2,x_3)+P(a_1,1,1,a_2,a_3|x_1,y_1,y_2,x_2,x_3)\\&-P(a_1,0,1,a_2,a_3|x_1,y_1,y_2,x_2,x_3)-P(a_1,1,0,a_2,a_3|x_1,y_1,y_2,x_2,x_3)].\end{aligned} \tag{A5}$$

Now, Eq. (A4) implies

$$P(a_1,i,j,a_2,a_3|x_1,y_1,y_2,x_2,x_3)=\sum_{\mathcal{D}}\delta_{i,s_{y_1}}\delta_{j,s_{y_2}}P_{18}(b_{10},b_{11},b_{12},b_{20},b_{21},b_{22},a_2,a_3|x_1,x_2,x_3),\quad\forall\, i,j\in\{0,1\}.$$

Using the above relations, in Eq. (A5) and $\mathcal{C}=\{a_1,a_2,a_3,b_{10},b_{11},b_{12},b_{20},b_{21},b_{22}\}$ we get

$$\begin{aligned}\langle\mathcal{A}_{1,x_1}\mathcal{B}_{1,y_1}\mathcal{B}_{2,y_2}\mathcal{A}_{2,x_2}\mathcal{A}_{3,x_3}\rangle&=\sum_{\mathcal{C}}(-1)^{a_1+a_2+a_3}\sum_{i,j=0,1}(-1)^{i+j}\delta_{i,s_{y_1}}\delta_{j,s_{y_2}}P_{18}(a_1,b_{10},b_{11},b_{12},b_{20},b_{21},b_{22},a_2,a_3|x_1,x_2,x_3)\\&=\sum_{\mathcal{C}}(-1)^{a_1+a_2+a_3+s_{y_1}+s_{y_2}}P_{18}(a_1,b_{10},b_{11},b_{12},b_{20},b_{21},b_{22},a_2,a_3|x_1,x_2,x_3).\end{aligned}$$

Hence,

$$\langle\mathcal{A}_{1,x_1}\mathcal{B}_{1,y_1}\mathcal{B}_{2,y_2}\mathcal{A}_{2,x_2}\mathcal{A}_{3,x_3}\rangle=\langle A_{1,x_1}B_1^{y_1}B_2^{y_2}A_{2,x_2}A_{3,x_3}\rangle. \tag{A6}$$

By Eqs. (A1), (A2), (A3), and(A6), we get the required criteria given by Eq. (15).

APPENDIX B

As already mentioned in the main text that distribution of qubits among the extreme parties is crucial in the context of obtaining violation by biseparable entanglement. So, for designing the protocol for purpose of detecting tripartite entanglement, all possible arrangements of qubits among the extreme parties are considered. At this junction, one may recall that in the nonlinear trilocal network scenario (Fig. 3), for a fixed source, the pattern of arranging qubits among the intermediate parties does not contribute in detecting the nature of biseparable entanglement. So, distribution of qubits only among the extreme parties $\mathcal{P}_1^E$, $\mathcal{P}_2^E$, and $\mathcal{P}_3^E$ is enlisted in Table VI. The last three columns of Table VI indicate the possible nature of biseparable entanglement of the unknown state under the circumstance that violation of at least one trilocal inequality [Eq. (15)] is obtained in the corresponding phase. For instance, consider the phase $t_{1,2,3}$. If violation is obtained in this phase of the protocol, then following are the possible natures of the three unknown quantum states:

(i) κ_1 is either genuinely entangled or have biseparable entanglement content in $12/3$ or $13/2$ cut.

(ii) κ_2 is either genuinely entangled or have biseparable entanglement content in $12/3$ or $23/1$ cut.

(iii) κ_3 is either genuinely entangled or have biseparable entanglement content in $23/1$ or $13/2$ cut.

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