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Any two-qubit state has nonzero quantum discord under global unitary operations

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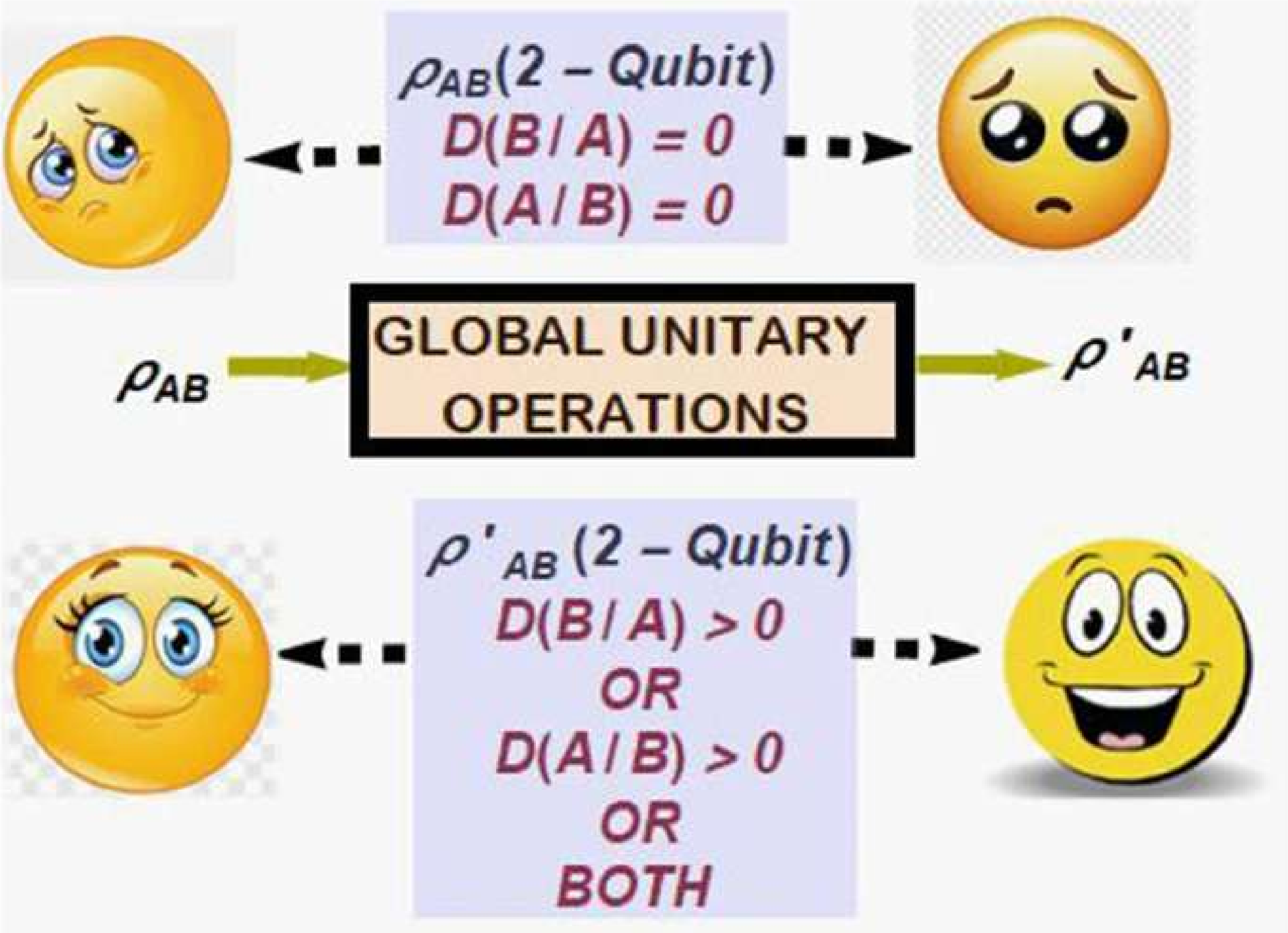
Abstract

Quantum discord is significant in analyzing quantum nonclassicality beyond the paradigm of entanglement. Presently, we have explored the effectiveness of global unitary operations in manifesting quantum discord from a general two-qubit zero discord state. Apart from the emergence of some obvious concepts such as absolute classical-quantum and absolute quantum-classical states, more interestingly, it is observed that set of states characterized by absoluteness contains only maximally mixed state. Consequently, this marks the peak of effectiveness of global unitary operations in purview of manifesting nonclassicality from arbitrary two-qubit state when other standard methods fail to do so. A set of effective global unitaries has been provided in this context. Our observations have direct implications in remote state preparation task.

Graphic Abstract



Data Availability Statement

This manuscript has associated data in a data repository. [Authors' comment: This manuscript has associated data in arxiv.org(quant-ph). Arxiv number is: arXiv:2004.12991(quant-ph).]

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Contributions

K. Mukherjee developed main idea of the work, performed the analysis and wrote the paper. S. Karmakar and B. Paul cross checked the findings and also assisted K. Mukherjee in writing the paper. All the authors have read and approved the final manuscript.

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Appendices

Appendix A

Proof of Theorem.1

Let $\rho_{AB} \in CQ$. Let it be subjected to an arbitrary general (global) unitary operation $\mathcal{U}$. It has already been discussed before that application of a general (global) unitary $\mathcal{U}$ over any two-qubit state can be interpreted as that of applying local unitary operations (on subsystems) followed by nonlocal unitary operations on the whole system and then again followed by local unitary operations. Let $\rho^{(1)}_{\mathcal{AB}}$, $\rho^{(2)}_{\mathcal{AB}}$, $\rho^{(3)}_{\mathcal{AB}}$ denote the transformed states in subsequent stages of transformation $\rho_{\mathcal{AB}} \rightarrow \rho^{'}_{\mathcal{AB}}$:

$$\begin{aligned}
\rho^{(1)}_{\mathcal{AB}} &= \mathcal{U}^1_{\mathcal{A}} \otimes \mathcal{U}^1_{\mathcal{B}} \rho_{\mathcal{AB}} (\mathcal{U}^1_{\mathcal{A}} \otimes \mathcal{U}^1_{\mathcal{B}})^\dagger \\
\rho^{(2)}_{\mathcal{AB}} &= \widehat{\mathcal{U}} \rho^{(1)}_{\mathcal{AB}} (\widehat{\mathcal{U}})^\dagger \\
\rho^{(3)}_{\mathcal{AB}} &= \rho^{'}_{\mathcal{AB}} = \mathcal{U}^2_{\mathcal{A}} \otimes \mathcal{U}^2_{\mathcal{B}} \rho^{(2)}_{\mathcal{AB}} (\mathcal{U}^2_{\mathcal{A}} \otimes \mathcal{U}^2_{\mathcal{B}})^\dagger
\end{aligned}$$
(23)

Now, applying local unitaries has no effect on quantum discord of a two-qubit state [18]. So if $\rho_{\mathcal{AB}} \in CQ$, then $\rho^{(1)}_{\mathcal{AB}} \in CQ$. Now, as $\rho^{(1)}_{\mathcal{AB}}$ is a classical-quantum state (Eq. 12), all possible forms (Bloch vector representation [29]) of $\rho^{(1)}_{\mathcal{AB}}$ are given in Table 1. Nonlocal unitary operation $\widehat{\mathcal{U}}$ is now applied on $\rho^{(1)}_{\mathcal{AB}}$. The detailed analysis of applying $\widehat{\mathcal{U}}$ on all possible forms of $\rho^{(1)}_{\mathcal{AB}}$ (to be discussed below) shows that for every possible form of $\rho^{(1)}_{\mathcal{AB}}$ except $\frac{1}{4}\mathbb{I}_{2\times2}$, there exists a nonlocal unitary operation (see Table 2 for suitable values of parameters ϕ_1, ϕ_2, ϕ_3) such that resulting state $\rho^{(2)}_{\mathcal{AB}}$ is not a classical-quantum state. Now, $\mathbb{D}(\mathcal{B}/\mathcal{A}) \neq 0$ if and only if $\rho^{(2)}_{\mathcal{AB}}$ is not a classical-quantum state [18]. Hence, every $\rho^{(2)}_{\mathcal{AB}}$ except $\frac{1}{4}\mathbb{I}_{2\times2}$ has nonzero 'one-way' discord ($\mathbb{D}(\mathcal{B}/\mathcal{A}) \neq 0$. Lastly, local unitary operation $\mathcal{U}^2_{\mathcal{A}} \otimes \mathcal{U}^2_{\mathcal{B}}$ is applied resulting in state $\rho^{(3)}_{\mathcal{AB}}$. $\mathbb{D}(\mathcal{B}/\mathcal{A})$ remaining invariant under local unitaries, and any possible form of $\rho^{(3)}_{\mathcal{AB}}$ except $\frac{1}{4}\mathbb{I}_{2\times2}$ has nonzero 'one-way' discord. *Hence*, excepting the maximally mixed state ($\frac{1}{4}\mathbb{I}_{2\times2}$), any member $\rho_{\mathcal{AB}}$ from the set of classical-quantum states (CQ) gets transformed into a 'one-way' nonzero discord state $\rho'_{\mathcal{AB}}$. Consequently, $ACQ = \{\frac{\mathbb{I}_{2\times 2}}{4}\}$ □

Analysis of the effect of nonlocal unitary operations $\widehat{\mathcal{U}}$ *on state* $\rho^{(1)}_{\mathcal{AB}}$ (Eq. 23): This part of the discussion is based on the necessary and sufficient condition that $\mathbb{D}_{\rho^{(2)}_{\mathcal{AB}}}(\mathcal{B}/\mathcal{A})$ vanishes if and only if it can be expressed as a classical-quantum state (Eq. 12). As indicated in the main text, every possible form of $\rho^{(1)}_{\mathcal{AB}}$ as a classical-quantum state is given in Table 1. Now, for each of those forms, if possible, let us assume that $\rho^{(2)}_{\mathcal{AB}}$ (Eq. 23) can be expressed as a classical-quantum state (Eq. 12). To be precise, we assume existence of unit vector $\mathbf{u} = (u_1, u_2, u_3)$ giving direction of projector $(\Pi^{\mathcal{A}}_i)$ corresponding to classical part of $\rho^2_{\mathcal{AB}}$. Now, under this assumption, $\forall i, k, j, l \in \{0, 1\}$ coefficient of $|ik\rangle\langle jl|$ of $\rho^{(2)}_{\mathcal{AB}}$, C_{ikjl}(say) should be equal to that of coefficient of $|ik\rangle\langle jl|$ corresponding to classical-quantum state form of $\rho^{(2)}_{\mathcal{AB}}$, C^{CQ}_{ikjl} (say). Given a $\rho^{(2)}_{\mathcal{AB}}$, failing to obtain equality ($C_{ikjl} = C^{CQ}_{ikjl}$) for at least one (i, k, j, l) indicates that such a comparison is impossible which in turn proves that our assumption is wrong: $\rho^{(2)}_{\mathcal{AB}}$ is not a classical-quantum state. Consequently, $\mathbb{D}_{\rho^{(2)}_{\mathcal{AB}}}(\mathcal{B}/\mathcal{A})$ turns out to be nonzero.

Now, Table 1 indicates two possible forms of classical quantum states (Eq. 12).

$$\rho^{(1)}_{\mathcal{AB}} = \frac{1}{4}(\mathbb{I}_{2\times 2} + \mathfrak{m}.\sigma \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathfrak{n}.\sigma)$$

(24)

and

$$\rho_{\mathcal{AB}}^{(1)} = \frac{1}{4}(\mathbb{I}_{2\times 2} + \mathfrak{m}_i\sigma_i \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathfrak{n}.\sigma + \mathfrak{s}_{ii}\sigma_i \otimes \sigma_i)$$
(25)

where $\mathfrak{m} = (\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3)$ and $\mathfrak{n} = (\mathfrak{n}_1, \mathfrak{n}_2, \mathfrak{n}_3)$ are real vectors. In Eq. (25), the index $i \in \{1, 2, 3\}$. Corresponding possible cases are as follows:

1. 1. $\mathfrak{m} = (m_1, 0, 0), \mathcal{S} = diag(s_{11}, 0, 0)$, $\mathfrak{n}$ arbitrary
2. 2. $\mathfrak{m} = (0, m_2, 0), \mathcal{S} = diag(0, s_{22}, 0)$, $\mathfrak{n}$ arbitrary
3. 3. $\mathfrak{m} = (0, 0, m_3), \mathcal{S} = diag(0, 0, s_{33})$, $\mathfrak{n}$ arbitrary

We now start with the first form (Eq. 24). For arbitrary $\mathfrak{n}$, depending on $\mathfrak{m}$ in Eq. (24) following cases are possible:

1. 1. $\mathfrak{m} = \boldsymbol{\Theta}$
2. 2. $\mathfrak{m} = (0, 0, m_3)$
3. 3. $\mathfrak{m} = (0, m_2, 0)$
4. 4. $\mathfrak{m} = (m_1, 0, 0)$
5. 5. $\mathfrak{m} = (m_1, 0, m_3)$
6. 6.

$\mathfrak{m} = (m_1, m_2, 0)$

7. 7.
   $\mathfrak{m} = (0, m_2, m_3)$

8. 8.
   $\mathfrak{m} = (m_1, m_2, m_3)$

Firstly, let us consider the trivial subcase of Case.1 where both $\mathfrak{m}$ and $\mathfrak{n} = \Theta$. This corresponds to the maximally mixed state:$\rho^{(1)}_{\mathcal{AB}} = \frac{1}{4}\mathbb{I}_{2\times 2}$. Clearly, after application of any nonlocal unitary operation, $\rho^{(1)}_{\mathcal{AB}}$ remains unchanged. Consequently, $\rho^{(2)}_{\mathcal{AB}}$ in this case is a classical-quantum state, thereby having vanishing discord.

We now approach with all possible nontrivial subcases related to each of the above cases starting with that of Case.1.

Case1: $\mathfrak{m} = \boldsymbol{\Theta}$ and $vec\mathfrak{n}$ is arbitrary. Possible subcases of Case.1 are:

- $\mathfrak{n} = (n_1, 0, 0)$
- $\mathfrak{n} = (0, n_2, 0)$
- $\mathfrak{n} = (n_1, n_2, 0)$
- $\mathfrak{n} = (n_1, n_2, n_3)$ with $n_3 \neq 0$ whereas n_1 and n_2 are arbitrary.

Subcase 1: Let nonlocal unitary operation $\widehat{\mathcal{U}} = \widehat{\mathcal{U}}(\phi_1, \phi_2, \phi_3)$ characterized by $\phi_1 = \frac{\pi}{2}$, $\phi_2 = \phi_3 = \frac{\pi}{4}$ be applied on $\rho^{(1)}_{\mathcal{AB}}$. As stated above, let us now consider coefficient of term $|01\rangle\langle 00|$ of $\rho^{(2)}_{\mathcal{AB}}(C_{0100})$ and that of coefficient of term $|01\rangle\langle 00|$ appearing in assumed classical-quantum state form of $\rho^{(2)}_{\mathcal{AB}}(C^{CQ}_{0100})$. Equality $C_{0100} = C^{CQ}_{0100}$ demands:

$$n_1(1-u_3^2) = 0$$
(26)

As $n_1 \neq 0$ and $u_1^2 + u_2^2 + u_3^2 = 1$, $u_3 = \pm 1$, $u_2 = u_1 = 0$. Again $C_{1000} = C^{CQ}_{1000}$ demands:

$$n_1(-1 + u_1 u_2 + u_2^2) = 0$$
(27)

Using $u_2 = u_1 = 0$ in Eq. (27) demands $n_1 = 0$ leading to a contradiction. Hence, $C_{0100} = C^{CQ}_{0100}$ and $C_{1000} = C^{CQ}_{1000}$ do not hold simultaneously. Consequently, for this subcase, $\rho^{(2)}_{\mathcal{AB}}$ obtained from classical-quantum state $\rho^{(1)}_{\mathcal{AB}}$ after applying nonlocal unitary operation $\widehat{\mathcal{U}}(\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{4})$ is not a classical-quantum state. So under application of $\widehat{\mathcal{U}}(\frac{\pi}{2},\frac{\pi}{4},\frac{\pi}{4})$, the classical-quantum state $\rho^{(1)}_{\mathcal{AB}}$ (Eq. 24), characterized by $\mathfrak{n} = (n_1, 0, 0)$ and $\mathfrak{m} = \Theta$, gets converted to a 'one-way' discord nonzero state.

Subcase 2: Let nonlocal unitary operation $\widehat{\mathcal{U}}(\frac{\pi}{4},0,\frac{\pi}{2})$ be applied. $C_{0100} = C^{CQ}_{0100}$ demands:

$$n_2(1-u_3^2) = 0$$
(28)

Hence, $C_{0100} = C^{CQ}_{0100}$ demands $u_3 = \pm 1$, $u_2 = u_1 = 0$. But $C_{1101} = C^{CQ}_{1101}$ demands:

$$n_2(1-u_2^2) = 0$$
(29)

which requires $n_2 = 0$ as u_2 must be 0 if $C_{0100} = C^{CQ}_{0100}$ (Eq. 28). This again leads to a contradiction as for this subcase, $n_2 \neq 0$.

Subcase 3: $\mathfrak{n} = (n_1, n_2, 0)$

Let nonlocal unitary operation $\widehat{\mathcal{U}}(\frac{\pi}{4}, 0, \frac{\pi}{2})$ be applied. $C_{1000} = C_{1000}^{CQ}$ demands:

$$n_2(1 - u_2^2) = 0$$

(30)

Hence, $C_{1000} = C_{1000}^{CQ}$ demands $u_2 = \pm 1$, $u_3 = u_1 = 0$. But $C_{0100} = C_{0100}^{CQ}$ demands:

$$n_1(1 - u_3^2) = 0$$

(31)

which requires $n_1 = 0$ as u_3 must be 0 which leads to a contradiction as for this subcase, $n_1 \neq 0$.

Subcase 4: $\mathfrak{n} = (n_1, n_2, n_3)$ with $n_3 \neq 0$ whereas n_1 and n_2 are arbitrary.

Let nonlocal unitary operation $\widehat{\mathcal{U}}(\frac{\pi}{4}, \frac{\pi}{4}, 0)$ be applied. $C_{0100} = C_{0100}^{CQ}$ demands:

$$u_3 u_1 n_3 = 0 \, and \, u_3 u_2 n_3 = 0$$

(32)

For the above relation to be true, if possible let $u_3 = 0$. But for $u_3 = 0$, $C_{0000} = C_{0000}^{CQ}$ demands $n_3 = 0$ which is a contradiction. So $u_3 \neq 0$. Consequently, Eq. (32) requires $u_1 = u_2 = 0$. But then $C_{1001} = C_{1001}^{CQ}$ demands $n_3 = 0$ which is impossible and so again contradiction obtained as for this subcase, $n_3 \neq 0$.

Case 2: $\mathfrak{m} = (0, 0, m_3)$.

Possible subcases are as follows:

Subcase 1: $\mathfrak{n} = (n_1, n_2, n_3)$ where $n_1 \neq 0$ and n_2, n_3 are arbitrary

Here, $\widehat{\mathcal{U}}(0, \frac{\pi}{2}, 0)$ is applied. $C_{1000} = C_{1000}^{CQ}$ demands:

$$n_1(1 - u_2^2) = 0$$
(33)

So, $u_3 = u_1 = 0$ and $u_2 = \pm 1$. But using these in relation $C_{1100} = C_{1100}^{CQ}$ gives $m_3 = 0$ which is impossible.

Subcase 2: $\mathfrak{n} = (0, n_2, n_3)$ where $n_2 \neq 0$ and n_3 is arbitrary. Here, $\widehat{\mathcal{U}}(0, \frac{\pi}{2}, \frac{\pi}{2})$ is applied. $C_{1100} = C_{1100}^{CQ}$ and $C_{1001} = C_{1001}^{CQ}$ demand:

$$m_3(u_3^2 + u_2^2) = 0$$
(34)

As $m_3 \neq 0$, $u_3 = u_2 = 0$ and $u_1 = \pm 1$. But using these in relation $C_{0100} = C_{0100}^{CQ}$ gives $n_2 = 0$ which leads to contradiction.

Subcase 3: $\mathfrak{n} = (0, 0, n_3)$ where n_3 is arbitrary $\widehat{\mathcal{U}}(0, \frac{\pi}{4}, \frac{\pi}{4})$ is applied. $C_{0000} = C_{0000}^{CQ}$ requires:

$$m_3(1 - u_3^2) = 0$$
(35)

So, $u_1 = u_2 = 0$ and $u_3 = \pm 1$. But using these in relations $C_{1001} = C_{1001}^{CQ}$ gives

$$n_3 - m_3 = 0$$
(36)

and in $C_{1100} = C_{1100}^{CQ}$ gives

$$n_3 + m_3 = 0$$

(37)

Above two relations require $m_3 = n_3 = 0$ which is impossible.

Case3: $\mathfrak{m} = (0, m_2, 0)$. Possible subcases are as follows:

Subcase 1:$\mathfrak{n} = (n_1, n_2, n_3)$ where $n_1 \neq 0$ and n_2, n_3 are arbitrary. Here, $\widehat{\mathcal{U}}(0, 0, \frac{\pi}{2})$ is applied. $C_{0100} = C_{0100}^{CQ}$ demands:

$$n_1(1 - u_3^2) = 0$$

(38)

So, $u_2 = u_1 = 0$ and $u_3 = \pm 1$. But using these in relation $C_{1000} = C_{1000}^{CQ}$ gives $m_2 = 0$ which is impossible as $m_2 \neq 0$.

Subcase 2: $\mathfrak{n} = (0, n_2, n_3)$ where $n_2 \neq 0$ and n_3 is arbitrary. $\widehat{\mathcal{U}}(0, \frac{\pi}{2}, \frac{\pi}{4})$ is applied.

$C_{0000} = C_{0000}^{CQ}$ demands:

$$m_2 u_3 (u_1 + u_2) = 0$$

(39)

As $m_2 \neq 0$, above relation implies either $u_3 = 0$ or $u_1 + u_2 = 0$. If possible, let $u_3 = 0$. Then, $C_{0100} = C_{0100}^{CQ}$ implies $n_2 = 0$ which is impossible. Hence, $u_3 \neq 0$. Consequently, $u_1 + u_2 = 0$. Now, using $u_3 \neq 0$ in $C_{0101} = C_{0101}^{CQ}$ gives $u_1 - u_2 = 0$.

Now, $u_1 \pm u_2 = 0$ implies $u_1 = u_2 = 0$. Now, this relation when used in $C_{0010} = C_{0010}^{CQ}$ gives $m_2 = 0$ which is impossible. Hence, Eq. (39) is impossible.

Subcase 3: $\mathfrak{n} = (0, 0, n_3)$ where n_3 is arbitrary. $\widehat{\mathcal{U}}(\frac{\pi}{4}, 0, \frac{\pi}{4})$ is applied. $C_{0100} = C_{0100}^{CQ}$ requires:

$$m_2(1 - u_3^2) = 0$$
(40)

So $u_2 = u_1 = 0$ and $u_3 = \pm 1$. But using these in relation $C_{1000} = C_{1000}^{CQ}$ gives $m_2 = 0$ which is impossible.

Case 4: $\mathfrak{m} = (m_1, 0, 0)$. Possible subcases are as follows:

Subcase 1: $\mathfrak{n} = (n_1, n_2, n_3)$ where $n_1 \neq 0$ and n_2, n_3 are arbitrary. Here, $\widehat{\mathcal{U}}(0, \frac{\pi}{2}, 0)$ is applied. $C_{0100} = C_{0100}^{CQ}$ demands:

$$m_1(1 - u_3^2) = 0$$
(41)

As $m_1 \neq 0$, $u_2 = u_1 = 0$ and $u_3 = \pm 1$. But using these in relation $C_{1000} = C_{1000}^{CQ}$ gives $n_1 = 0$ which is impossible as $n_1 \neq 0$.

Subcase 2: $\mathfrak{n} = (0, n_2, n_3)$ where $n_2 \neq 0$ and n_3 is arbitrary. $\widehat{\mathcal{U}}(\frac{\pi}{4}, \frac{\pi}{2}, 0)$ is applied.

$C_{0100} = C_{0100}^{CQ}$ demands relation given by Eq. (41). So $u_2 = u_1 = 0$ and $u_3 = \pm 1$. But using these in relation $C_{1000} = C_{1000}^{CQ}$ gives $n_2 = 0$ which is impossible as $n_2 \neq 0$.

Subcase 3: $\mathfrak{n} = (0, 0, n_3)$ where n_3 is arbitrary. $\widehat{\mathcal{U}}(0, \frac{\pi}{4}, \frac{\pi}{4})$ is applied.

$C_{0100} = C_{0100}^{CQ}$ demands relation given by Eq. (41) and hence $u_2 = u_1 = 0$ and $u_3 = \pm 1$. But using these in relation $C_{1000} = C_{1000}^{CQ}$ gives $m_1 = 0$ which is impossible.

Case 5: $\mathfrak{m} = (m_1, 0, m_3)$ and $\mathfrak{n}$ is arbitrary.
$\widehat{\mathcal{U}}(0, 0, \frac{\pi}{2})$ is applied.

$C_{0000} = C_{0000}^{CQ}$ and $C_{0101} = C_{0101}^{CQ}$ together demand:

$$m_3(1 - u_3^2) = 0$$
(42)

As $m_3 \neq 0$, $u_2 = u_1 = 0$ and $u_3 = \pm 1$. But using these in relation $C_{1000} = C_{1000}^{CQ}$ gives $m_1 = 0$ which is impossible as $m_1 \neq 0$.

Case 6: $\mathfrak{m} = (m_1, m_2, 0)$ and $\mathfrak{n}$ is arbitrary.
$\widehat{\mathcal{U}}(0, \frac{\pi}{2}, 0)$ is applied.

$C_{1011} = C_{1011}^{CQ}$ demands relation given by Eq. (41). As $m_1 \neq 0$, $u_2 = u_1 = 0$ and $u_3 = \pm 1$. But using these in relation $C_{0111} = C_{0111}^{CQ}$ gives $m_2 - n_1 = 0$ and in relation $C_{0010} = C_{0010}^{CQ}$ gives $m_2 + n_1 = 0$. This in turn implies $m_2 = n_1 = 0$ which is impossible as $m_2 \neq 0$.

Case 7: $\mathfrak{m} = (0, m_2, m_3)$ and $\mathfrak{n}$ is arbitrary.
$\widehat{\mathcal{U}}(\pi, \pi, \frac{\pi}{2})$ is applied.

$C_{0000} = C_{0000}^{CQ}$ and $C_{0101} = C_{0101}^{CQ}$ together demand relation given by Eq. (42). As $m_3 \neq 0$, $u_2 = u_1 = 0$ and $u_3 = \pm 1$. But using these in relation $C_{1101} = C_{1101}^{CQ}$ gives $m_2 = 0$ which is impossible as $m_2 \neq 0$.

Case 8: $\mathfrak{m} = (m_1, m_2, m_3)$ and $\mathfrak{n}$ is arbitrary.
Clearly, this case can be proved by anyone of above

three cases (5–7).

Now, we consider the possible cases of states given by Eq. (25).

Case 1.: $\mathfrak{m} = (m_1, 0, 0)$, $\mathcal{S} = diag(s_{11}, 0, 0)$ and $\mathfrak{n}$ arbitrary. $\widehat{\mathcal{U}}(0, 0, \frac{\pi}{2})$ is applied. $C_{0111} = C^{CQ}_{0111}$ requires:

$$m_1(1 - u_2^2) = 0$$
(43)

So $u_3 = u_1 = 0$ and $u_2 = \pm 1$. But using these in relation $C_{0011} = C^{CQ}_{0011}$ gives $s_{11} = 0$ which is impossible.

Case 2.: $\mathfrak{m} = (0, m_2, 0)$, $\mathcal{S} = diag(0, s_{22}, 0)$, $\mathfrak{n}$ arbitrary. $\widehat{\mathcal{U}}(\frac{\pi}{2}, \frac{\pi}{2}, 0)$ is applied. $C_{1011} = C^{CQ}_{1011}$ requires relation given by Eq. (40) and hence $u_2 = u_1 = 0$ and $u_3 = \pm 1$. But using these in relation $C_{0011} = C^{CQ}_{0011}$ gives $s_{22} = 0$ which is impossible.

Case 3.: $\mathfrak{m} = (0, 0, m_3)$, $\mathcal{S} = diag(0, 0, s_{33})$, $\mathfrak{n}$ arbitrary. $\widehat{\mathcal{U}}(\pi, \frac{\pi}{2}, \pi)$ is applied. $C_{0011} = C^{CQ}_{0011}$ and $C_{0110} = C^{CQ}_{0110}$ together require relation given by Eq. (34) and hence $u_3 = u_1 = 0$ and $u_1 = \pm 1$. But using these in relation $C_{1111} = C^{CQ}_{1111}$ gives $s_{33} = 0$ which is impossible. So in each of the possible cases of $\rho^{(1)}_{\mathcal{AB}}$, it is shown that after applying suitable nonlocal unitary operation, the transformed state $\rho^{(2)}_{\mathcal{AB}}$ no longer remains a classical-quantum state. Consequently, $\mathbb{D}_{\rho^{(2)}_{\mathcal{AB}}}(\mathcal{B}/\mathcal{A}) \neq 0$. We enlist the suitable required nonlocal unitary operations for all possible subcases of individual cases corresponding to first possible form of $\rho^{(1)}_{\mathcal{AB}}$ (Eq. 24) and also for second possible form given by Eq. (25).

Appendix B

Here, we discuss the effect of nonlocal unitary operations over all possible forms of quantum-classical states [29]. As discussed in 'Appendix A,' here also we enlist those nonlocal unitaries which are effective in generating states having nonvanishing $\mathbb{D}_{\rho'_{\mathcal{AB}}}(\mathcal{A}/\mathcal{B})$ starting from quantum-classical states $\rho_{\mathcal{AB}}$. One of the forms of quantum-classical state (after application of suitable local unitaries) is given by Eq. (24), while the other is given by:

$$\rho^{(1)}_{\mathcal{AB}} = \frac{1}{4}(\mathbb{I}_{2\times2} + \mathfrak{m}.\sigma \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathfrak{n}_i\sigma_i + \mathfrak{s}_{ii}\sigma_i \otimes \sigma_i)(i = 1, 2, 3)$$

(44)

With eight possible forms (as listed in 'Appendix A') of quantum-classical states corresponding to Eq. (24), the possible cases as given by Eq. (44) are:

1. 1. $\mathfrak{n} = (n_1, 0, 0)$, $\mathcal{S} = diag(s_{11}, 0, 0)$, $\mathfrak{m}$ arbitrary

2. 2. $\mathfrak{n} = (0, n_2, 0)$, $\mathcal{S} = diag(0, s_{22}, 0)$, $\mathfrak{m}$ arbitrary

3. 3. $\mathfrak{n} = (0, 0, n_3)$, $\mathcal{S} = diag(0, 0, s_{33})$, $\mathfrak{m}$ arbitrary

We now enlist the effective nonlocal unitaries for all possible cases in Table 3.

Appendix C

Discussing effectiveness of global unitary operations to convert zero discord state to nonzero discord state in main text, here we discuss the mechanism of this conversion in detail.

Application of Global Unitary Operations: Given an arbitrary two-qubit state $\rho_{\mathcal{AB}}$, with correlation tensor $\mathcal{T}$, singular value decomposition of $\mathcal{T}$ may be obtained by performing suitable local unitary operations $\mathcal{U}_{\mathcal{A}}^{1} \otimes \mathcal{U}_{\mathcal{B}}^{1}$ over $\rho_{\mathcal{AB}}$ [43]. Let $\rho_{\mathcal{AB}}$ be a classical-quantum state. Let $\kappa_i^L (i = 1, 2, 3)$ and , $\kappa_i^R (i = 1, 2, 3)$ denote orthonormalized left and right singular vectors of $\mathcal{T}$, respectively. $\forall i$, denoting $\kappa_i^{L(R)} \in \mathbb{R}^3$ as $(\kappa_{i1}^{L(R)}, \kappa_{i2}^{L(R)}, \kappa_{i3}^{L(R)})$, local unitary matrices $\mathcal{U}_{\mathcal{A}}^{1}, \mathcal{U}_{\mathcal{B}}^{1}$ are given by:

$$\mathcal{U}_{\mathcal{A}}^{1} = \begin{pmatrix} \kappa_{11}^L & \kappa_{12}^L & \kappa_{13}^L \\ \kappa_{21}^L & \kappa_{22}^L & \kappa_{23}^L \\ \kappa_{31}^L & \kappa_{32}^L & \kappa_{33}^L \end{pmatrix}$$
$$\mathcal{U}_{\mathcal{B}}^{1} = \begin{pmatrix} \kappa_{11}^R & \kappa_{12}^R & \kappa_{13}^R \\ \kappa_{21}^R & \kappa_{22}^R & \kappa_{23}^R \\ \kappa_{31}^R & \kappa_{32}^R & \kappa_{33}^R \end{pmatrix}$$
(45)

$\rho_{\mathcal{AB}}$, being a classical-quantum state, after application of the local unitary operations $\mathcal{U}_{\mathcal{A}}^{1} \otimes \mathcal{U}_{\mathcal{B}}^{1}$ (Eq. 45), $\rho_{\mathcal{AB}}^{(1)}$ (Eq. 23) corresponds to one of the possible forms prescribed in Table 1. Then, observing the exact form of $\rho_{\mathcal{AB}}^{(1)}$ the suitable nonlocal unitary operation $\widehat{\mathcal{U}}(\phi_1, \phi_2, \phi_3)$ to be applied is chosen from Table 2 (which enlists required nonlocal unitary for every possible form of classical-quantum state given in Table 1).

Table 2 Details of nonlocal unitary operations to be applied on any possible classical-quantum state having forms given by Eqs. (24) and (25) so that resulting state has nonzero $\mathbb{D}_{\rho'_{\mathcal{AB}}}(\mathcal{B}/\mathcal{A})$

Table 3 List of suitable nonlocal unitary operations application of which converts any possible quantum-classical state (forms given by Eqs. (24), (44)) to $\rho'_{\mathcal{AB}}$ such that $\mathbb{D}_{\rho'_{\mathcal{AB}}}(\mathcal{A}/\mathcal{B})$

Now, to obtain 'one-way' zero discord state starting from an arbitrary quantum-classical state $\rho_{\mathcal{AB}}$, analogous treatment is to be made with now considering Table 3 instead of for obvious reasons.

An Example: Consider a two-qubit product state:

$$\rho_{prod} = \frac{1}{4}(\mathbb{I}_2 + \mathbf{r}^a.\sigma) \otimes (\mathbb{I}_2 + \mathbf{r}^b.\sigma)$$
(46)

where $\mathbf{r}^{a(b)} = (r_1^{a(b)}, r_2^{a(b)}, r_3^{a(b)})$.

$\mathbb{D}_{\rho_{prod}}(\mathcal{B}/\mathcal{A}) = \mathbb{D}_{\rho_{prod}}(\mathcal{A}/\mathcal{B}) = 0$ [29]. So ρ_{prod} is a zero discord state. Correlation tensor is given by:

$$\mathcal{T}_{prod} = \begin{pmatrix} r_1^a r_1^b & r_1^a r_2^b & r_1^a r_3^b \\ r_2^a r_1^b & r_2^a r_2^b & r_2^a r_3^b \\ r_3^a r_1^b & r_3^a r_2^b & r_3^a r_3^b \end{pmatrix}$$
(47)

The suitable local unitary operations are:

$$\mathcal{U}^1_{\mathcal{A}(\mathcal{B})} = \begin{pmatrix} -\frac{r_3^{a(b)}}{r_1^{a(b)} n_1^{a(b)}} & 0 & \frac{1}{n_1^{a(b)}} \\ -\frac{r_2^{a(b)}}{r_1^{a(b)} n_2^{a(b)}} & \frac{1}{n_2^{a(b)}} & 0 \\ \frac{r_1^{a(b)}}{r_3^{a(b)} n_3^{a(b)}} & \frac{r_2^{a(b)}}{r_3^{a(b)} n_3^{a(b)}} & \frac{1}{n_3^{a(b)}} \end{pmatrix}$$

where $n_1^{a(b)} = \sqrt{1 + (\frac{r_3^{a(b)}}{r_1^{a(b)}})^2}$, $n_2^{a(b)} = \sqrt{1 + (\frac{r_2^{a(b)}}{r_1^{a(b)}})^2}$ and $n_3^{a(b)} = \sqrt{\frac{(r_1^{a(b)})^2 + (r_2^{a(b)})^2 + (r_3^{a(b)})^2}{(r_3^{a(b)})^2}}$. After application of these local unitary operations, $\rho^{(1)}_{prod}$ is given by:

$$\rho^{(1)}_{prod} = \frac{1}{4}(\mathbb{I}_{2\times 2} + \mathfrak{a}^{(1)}.\sigma \otimes \mathbb{I}_2 + \mathbb{I}_2 \otimes \mathfrak{b}^{(1)}.\sigma + \sum_{j_1,j_2=1}^{3} \mathfrak{t}^{(1)}_{j_1 j_2} \sigma_{j_1} \otimes \sigma_{j_2}),$$

(48)

where $\mathfrak{a}^{(1)} = (0, 0, n_3^a r_3^a)$, $\mathfrak{b}^{(1)} = (0, 0, n_3^b r_3^b)$ and correlation tensor $\mathcal{T}^{(1)}_{prod}$ is a diagonal matrix $diag(0, 0, r_3^a r_3^b n_3^a n_3^b)$. Clearly, $\rho^{(1)}_{prod}$ (Eq. 48) corresponds to a form in Table 1 and is also a quantum-classical state [29]. Observing the particular form in Table 1, on application of nonlocal unitary operation $\widehat{\mathcal{U}}(\pi, \frac{\pi}{2}, \pi)$ (as prescribed by Table 2), resulting state $\rho^{(2)}_{prod}$ is no longer a classical-quantum state ($\mathbb{D}_{\rho^{(2)}_{prod}}(\mathcal{B}/\mathcal{A}) > 0$). Again treating $\rho^{(1)}_{prod}$ as a quantum-classical state, $\widehat{\mathcal{U}}(\pi, \frac{\pi}{2}, 0)$ is the suitable nonlocal unitary operation (Table 3) application of which gives $\mathbb{D}_{\rho^{(2)}_{prod}}(\mathcal{A}/\mathcal{B}) > 0$. So $\rho^{(2)}_{prod}$ turns out to be a nonzero discord state. Lastly, suitable local unitaries $\mathcal{U}^2_{\mathcal{A}}, \mathcal{U}^2_{\mathcal{B}}$ may again be applied so as to obtain a simplified version of the state.

Alternatively, given any $\rho_{\mathcal{AB}}$ one may directly apply suitable nonlocal unitary operation $\widehat{\mathcal{U}}(\phi_1, \phi_2, \phi_3)$ considering $\mathcal{U}^1_{\mathcal{A}} = \mathcal{U}^1_{\mathcal{B}} = \frac{\mathbb{I}}{2}$. One such instance is cited in next section.

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